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DIRECTIONAL TIME–FREQUENCY ANALYSIS VIA CONTINUOUS FRAMES

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Abstract

Grafakos and Sansing [‘Gabor frames and directional time–frequency analysis’, *Appl. Comput. Harmon. Anal.* **25** (2008), 47–67] have shown how to obtain directionally sensitive time–frequency decompositions in $L^2(\mathbb{R}^n)$ based on Gabor systems in $L^2(\mathbb{R})$. The key tool is the ‘ridge idea’, which lifts a function of one variable to a function of several variables. We generalise their result in two steps: first by showing that similar results hold starting with general frames for $L^2(\mathbb{R})$, in the settings of both discrete frames and continuous frames, and second by extending the representations to Sobolev spaces. The first step allows us to apply the theory to several other classes of frames, for example wavelet frames and shift-invariant systems, and the second one significantly extends the class of examples and applications. We consider applications to the Meyer wavelet and complex B-splines. In the special case of wavelet systems we show how to discretise the representations using ϵ-nets.

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1. Introduction

Expansions of functions or signals as superpositions of basic building blocks with desired properties is one of the main tools in signal analysis. The expansions can be in terms of an integral, a discrete sum or a combination of both.

Many real-world signals depend on more than one variable. Depending on the type of expansion one is interested in, there are various ways to obtain such expansions. If an orthonormal basis for $L^2(\mathbb{R})$ is given, one can obtain an orthonormal basis for $L^2(\mathbb{R}^n)$ for all $n \in \mathbb{N}$ via a simple tensor product, but often a more flexible design is required. Some of the standard methods to obtain expansions in $L^2(\mathbb{R})$, for example wavelet frames or Gabor frames, have similar versions in $L^2(\mathbb{R}^n)$, but they might not be optimal for detecting features or special properties of the signal at hand. Other expansions are born in $L^2(\mathbb{R}^n)$, typically for $n = 2, 3$, for example caplets, ridgelets, curvelets

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and shearlets (see [4–6, 16, 20, 23] and the references therein); all of these can be considered as higher-dimensional wavelet-type systems with additional structure.

A different approach (parallel to the ridgelet construction) for Gabor systems was proposed by Grafakos and Sansing [17]. Starting with Gabor systems in $L^2(\mathbb{R})$, they developed a directionally sensitive Gabor-type expansion in $L^2(\mathbb{R}^n)$ using ridge functions. Two approaches were discussed in [17]: a discrete version, based on Gabor frames for $L^2(\mathbb{R})$, and a semidiscrete version based on continuous Gabor systems generated by two nonperpendicular functions.

In this paper, we extend the main results in [17] in various ways. First, we observe that the above-mentioned nonorthogonality places [17] in the setting of continuous frames, originally developed by Ali *et al.* [1] and Kaiser [21]. Using techniques from frame theory, we then prove that the results in [17] have parallel versions starting with general frames for $L^2(\mathbb{R})$ in both the discrete and the continuous settings. These results are directly applicable to the Meyer wavelet and complex B-splines. As a second step of generalisation we extend the decompositions to Sobolev spaces, an extension that significantly enlarges the class of available frame decompositions. In the special case of wavelet systems we also show how to discretise the representations using ϵ-nets.

In the rest of the introduction, we provide some notation and state the necessary facts about ridge functions and (continuous) frames. Then, in Section 2, we present the generalisations of the results in [17]. Semidiscrete representations of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ are investigated in Section 3, where we also apply the results to the Meyer wavelet and to complex B-splines. In the final Section 4, we obtain fully discrete representations for wavelet-type systems on bounded domains, by replacing the integral over the unit sphere by an appropriately chosen ε-net.

Some remarks concerning the notation: throughout the article, we assume that $n \in \mathbb{N}$. Since we deal with functions in $L^1(\mathbb{R})$ and lift them to functions in $L^2(\mathbb{R}^n)$, we need to consider inner products and the Fourier transform on different spaces. In general, for functions $f \in L^1(\mathbb{R}^n)$, $n \in \mathbb{N}$, we define the Fourier transform by

$$\widehat{f}(\gamma) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \gamma}\, d\gamma, \qquad \gamma \in \mathbb{R}^n,$$

where $x \cdot \gamma$ denotes the canonical inner product on $\mathbb{R}^n$.

We extend the Fourier transform to a unitary operator on $L^2(\mathbb{R}^n)$ in the usual way. The inverse Fourier transform of a function f will be denoted by $f^\vee$. Also, for functions $f, g : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, we use the notation

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx$$

whenever the right-hand side converges. The unit sphere in $\mathbb{R}^n$ will be denoted by $\mathbb{S}^{n-1}$. We also consider the Sobolev–Slobodeckij spaces, for $\alpha \geq 0$, defined by

$$H^\alpha(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \mid (1 + |\cdot|^2)^{\alpha/2}\, \widehat{f} \in L^2(\mathbb{R}^n)\}$$

(see, for example, [19, 30]). It is clear that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Let us now introduce the 'ridge procedure' that lifts functions of one variable to functions of several variables. Ridge functions were originally introduced by Pinkus [28]. Our starting point is to extend the ordinary differential operator on $\mathbb{R}$ to certain nondifferentiable functions. In fact, given $\beta > 0$, define the differential operator $\mathcal{D}^\beta$ acting on functions in the Schwartz space of rapidly decreasing functions, $h \in \mathcal{S}(\mathbb{R})$, or $h \in H^\alpha(\mathbb{R})$, $\alpha > \beta$, by

$$\mathcal{D}^\beta(h) := (\widehat{h}(\cdot)|\cdot|^\beta)^\vee.$$

Throughout, we use the following terminology, which relates functions (written with lower case letters), the corresponding ridge functions (written with a subscript), the action of the differential operator on the given function (written with capital letters) and the associated ridge function (written with capital letters and a subscript).

Definition 1.1. Consider any function $g : \mathbb{R} \to \mathbb{R}$.

(i) For $u \in \mathbb{S}^{n-1}$, define the ridge function $g_u : \mathbb{R}^n \to \mathbb{R}$ for g by

$$g_u(x) := g(u \cdot x), \quad x \in \mathbb{R}^n.$$

(ii) Whenever $g \in H^\alpha(\mathbb{R})$, $\alpha > 0$ or $g \in \mathcal{S}(\mathbb{R})$, let

$$G(s) := \mathcal{D}^{(n-1)/2} g(s), \quad s \in \mathbb{R}. \tag{1.1}$$

(iii) For $u \in \mathbb{S}^{n-1}$, define the weighted ridge function G_u for G by

$$G_u(x) := G(u \cdot x), \quad x \in \mathbb{R}^n.$$

Given $u \in \mathbb{S}^{n-1}$, the *Radon transform* of a function $f \in \mathcal{S}(\mathbb{R}^n)$ (in the direction u) is defined by

$$R_u f(s) := \int_{u \cdot x = s} f(x)\, dx, \quad s \in \mathbb{R}.$$

The Radon transform can be extended to a bounded operator from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R})$ [27, page 16 ff].

We note that the *Fourier slice theorem* relates the (one-dimensional) Fourier transform of the Radon transform of a function $f \in L^1(\mathbb{R}^n)$ to the (n-dimensional) Fourier transform of f by the formula

$$\widehat{R_u(f)}(\eta) = \widehat{f}(\eta u), \quad \eta \in \mathbb{R}, u \in \mathbb{S}^{n-1}.$$

In the following, we consider the Sobolev spaces $H^\alpha(\mathbb{R}^n)$ for $\alpha \geq 0$. In the one-dimensional case we will make use of the fact that $\mathcal{S}(\mathbb{R})$ is dense in $H^\alpha(\mathbb{R})$ for all $\alpha \geq 0$ (see [32, Theorem, Section 2.3.3] and [30, Ch. 8.8]). In addition, we require a result about the Radon transform R_u on Sobolev spaces.

Lemma 1.2. *Let $\alpha \geq 0$. For $u \in \mathbb{S}^{n-1}$, the Radon transform R_u is a linear isomorphism from $H^\alpha(\mathbb{R}^n)$ to $H^{\alpha+(n-1)/2}(\mathbb{R})$.*

Proof. By [27, Theorem 5.1 and Section VII.4], there exist constants $c_1, c_2 > 0$ (depending only on α and n) such that

$$c_1\|f\|_{H^\alpha(\mathbb{R}^n)} \leq \|R_u f\|_{H^{\alpha+(n-1)/2}(\mathbb{R})} \leq c_2\|f\|_{H^\alpha(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

The result now follows immediately since $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in every $H^\beta(\mathbb{R}^n)$, $\beta \geq 0$. □

The next lemma shows a close relation between ridge functions and the Radon transform.

Lemma 1.3. *Let $u \in \mathbb{S}^{n-1}$. Let either:*

(i) *$f \in L^1(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R})$; or*

(ii) *$f \in H^\alpha(\mathbb{R}^n)$ and $g \in H^\beta(\mathbb{R}^n)$, $\alpha, \beta \geq 0$.*

Then

$$\langle f, g_u\rangle = \langle R_u f, g\rangle.$$

Proof. First suppose that $g \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}\langle f, g_u\rangle &= \int_{\mathbb{R}^n} f(x)\overline{g(u\cdot x)}\,dx = \int_{-\infty}^{\infty}\Big(\int_{u\cdot x=s} f(x)\overline{g(u\cdot x)}\,dx\Big)\,ds\\ &= \int_{-\infty}^{\infty}\Big(\int_{u\cdot x=s} f(x)\,dx\Big)\overline{g(s)}\,ds = \langle R_u f, g\rangle,\end{aligned}$$

as desired.

In case (i), the right-hand side is finite, because $R_u f \in L^1(\mathbb{R}^n)$. In case (ii), the right-hand side is bounded, because of Lemma 1.2: $R_u f \in H^{\alpha+(n-1)/2}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R})$ is dense in $H^\beta(\mathbb{R}^n)$, the equality holds for all $g \in H^\beta(\mathbb{R}^n)$ because the right-hand side holds in the L^2-sense: $H^{\alpha+(n-1)/2}(\mathbb{R}^n), H^\beta(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. □

We will now state the necessary results about continuous frames. Recall that given a complex Hilbert space $\mathcal{H}$ and a measure space M with a positive measure μ, a *continuous frame* is a family of vectors $\{f_k\}_{k\in M}$ such that the mapping $k \mapsto \langle f, f_k\rangle$ is a measurable function on M for all $f \in \mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A\,\|f\|^2 \leq \int_M |\langle f, f_k\rangle|^2\,d\mu(k) \leq B\,\|f\|^2 \quad \forall f \in \mathcal{H}.$$

A continuous frame $\{f_k\}_{k\in M}$ is *tight* if we can choose $A = B$.

For every continuous frame, there exists at least one *dual continuous frame,* that is, a continuous frame $\{g_k\}_{k\in M}$ such that each $f \in \mathcal{H}$ has the representation

$$f = \int_M \langle f, f_k\rangle g_k\,d\mu(k); \tag{1.2}$$

the integral in (1.2) should be interpreted in the weak sense.

If $\{f_k\}_{k\in M}$ is a continuous tight frame with bound A, then $\{A^{-1}f_k\}_{k\in M}$ is a dual continuous frame.

Continuous frames generalise the more widely known (discrete) frames, which correspond to the case where M is a countable set equipped with the counting measure. Continuous frames were introduced independently by Ali *et al.* [1] and Kaiser [21].

There are well-known examples of continuous frames for $L^2(\mathbb{R})$ available in the literature. In order to introduce these, consider the translation, modulation and scaling operators on $L^2(\mathbb{R})$ defined by

$$T_a f(x) := f(x-a), \quad E_b f(x) := e^{2\pi i b x} f(x), \quad D_c f(x) := c^{1/2} f(cx),$$

where $a, b \in \mathbb{R}, c > 0$.

A system of functions of the form $\{E_b T_a g\}_{a,b\in\mathbb{R}}$ is called a (continuous) *Gabor system.* We need the following result, which is an easy consequence of standard results (see, for example, [7, Proposition 9.9.1] and [18, Theorem 3.2.1]).

Proposition 1.4.

(i) *For any $g \in L^2(\mathbb{R})\backslash\{0\}$, the Gabor system $\{E_b T_a g\}_{a,b\in\mathbb{R}}$ is a continuous tight frame for $L^2(\mathbb{R})$ with respect to $M = \mathbb{R}^2$ equipped with the Lebesgue measure, with frame bound $A = \|g\|^2$.*

(ii) *For any functions $g_1, g_2 \in L^2(\mathbb{R})$ for which $\langle g_1, g_2\rangle \neq 0$, the Gabor systems $\{E_b T_a g_1\}_{a,b\in\mathbb{R}}$ and $\{(1/\langle g_1, g_2\rangle) E_b T_a g_2\}_{a,b\in\mathbb{R}}$ are dual continuous frames.*

A *wavelet system* has the form $\{D_a T_b \psi\}_{a\neq 0, b\in\mathbb{R}}$ for a suitable function $\psi \in L^2(\mathbb{R})$. We say that ψ satisfies the *admissibility condition* if

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\gamma)|^2}{|\gamma|}\, d\gamma < \infty. \tag{1.3}$$

The admissibility condition together with [12, Proposition 2.4.1] immediately leads to a construction of a tight frame.

Corollary 1.5. *If $\psi \in L^2(\mathbb{R})$ is admissible, then $\{D_a T_b \psi\}_{a\neq 0, b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}$ equipped with the Haar measure $(1/a^2)\, da\, db$, with frame bound $A = C_\psi$.*

Note that the admissibility condition is easy to satisfy, even with generators $\psi \in \mathcal{S}(\mathbb{R})$. In fact, all functions $\psi \in \mathcal{S}(\mathbb{R})$ with vanishing mean,

$$\int_{\mathbb{R}} \psi(x)\, dx = \widehat{\psi}(0) = 0,$$

satisfy the admissibility condition (see, for example, [25, Ch. 1]).

2. Decompositions via continuous frames

We first show that any pair of continuous dual frames for $L^2(\mathbb{R})$ consisting of functions in $\mathcal{S}(\mathbb{R})$ leads to an integral representation of functions $f \in L^1(\mathbb{R}^n)$ for which $\widehat{f} \in L^1(\mathbb{R}^n)$. This generalises [17, Theorem 3], which interestingly does not use the term *continuous frame,* but just the technical condition $\langle g_1, g_2\rangle \neq 0$; in the particular context of Gabor analysis this means exactly that the functions g_1, g_2 generate continuous dual Gabor frames, as we saw in Corollary 1.4.

Theorem 2.1. *Let $\{f_k\}_{k\in M}$ and $\{g_k\}_{k\in M}$ be dual continuous frames for $L^2(\mathbb{R})$. If either:*

(i) $\{f_k\}_{k\in M}, \{g_k\}_{k\in M} \subset \mathcal{S}(\mathbb{R})$; *or*

(ii) $\{f_k\}_{k\in M}, \{g_k\}_{k\in M} \subset H^{\alpha+(n-1)/2}(\mathbb{R})$, $\alpha \geq 0$,

then, for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R})$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$,

$$f = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle f, G_{k,u}\rangle F_{k,u}\, dk\, du.$$

Proof. First note that for $g \in \mathcal{S}(\mathbb{R})$, we have $G \in \mathcal{S}(\mathbb{R})$ and, for $g \in H^{\alpha+(n-1)/2}(\mathbb{R})$, we have $G \in H^{\alpha}(\mathbb{R})$ for all $\alpha \geq 0$.

Consider case (i) first. Calculating the left-hand side using Lemma 1.3 yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle f, G_{k,u}\rangle F_{k,u}(x)\, dk\, du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle R_u f, G_k\rangle F_k(u\cdot x)\, dk\, du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle \widehat{R_u f}, \widehat{G_k}\rangle F_k(u\cdot x)\, dk\, du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \int_{-\infty}^{\infty} \widehat{R_u f}(\sigma)\overline{\widehat{g_k}(\sigma)}\, |\sigma|^{(n-1)/2}\, d\sigma F_k(u\cdot x)\, dk\, du.
\end{aligned}$$

Now,

$$F_k(u\cdot x) = (\widehat{f_k}(\cdot)|\cdot|^{(n-1)/2})^{\vee}(u\cdot x) = \int_{-\infty}^{\infty} e^{2\pi i\eta u\cdot x}\widehat{f_k}(\eta)|\eta|^{(n-1)/2}\, d\eta,$$

so we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle f, G_{k,u}\rangle F_{k,u}(x)\, dk\, du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \int_{-\infty}^{\infty} \widehat{R_u f}(\sigma)\overline{\widehat{g_k}(\sigma)}|\sigma|^{(n-1)/2}\, d\sigma \int_{-\infty}^{\infty} e^{2\pi i\eta u\cdot x}\widehat{f_k}(\eta)|\eta|^{(n-1)/2}\, d\eta\, dk\, du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}\Big(\int_M\int_{-\infty}^{\infty} |\sigma|^{(n-1)/2}\widehat{R_u f}(\sigma)\overline{\widehat{g_k}(\sigma)}\, d\sigma \widehat{f_k}(\eta)\, dk\Big) e^{2\pi i\eta u\cdot x}|\eta|^{(n-1)/2}\, d\eta\, du.
\end{aligned}$$

Note that because $\{f_k\}_{k\in M}$ and $\{g_k\}_{k\in M}$ are dual continuous frames for $L^2(\mathbb{R})$, also $\{\widehat{f_k}\}_{k\in M}$ and $\{\widehat{g_k}\}_{k\in M}$ are dual continuous frames. The term in the parentheses above, that is,

$$\int_M\int_{-\infty}^{\infty} |\sigma|^{(n-1)/2}\widehat{R_u f}(\sigma)\overline{\widehat{g_k}(\sigma)}\, d\sigma \widehat{f_k}(\eta)\, dk,$$

is exactly the frame decomposition with respect to these frames of the function $|\cdot|^{(n-1)/2}\widehat{R_u f}(\cdot)$ evaluated at the point η; thus,

$$\Big(\int_M\int_{-\infty}^{\infty} |\sigma|^{(n-1)/2}\widehat{R_u f}(\sigma)\overline{\widehat{g_k}(\sigma)}\, d\sigma \widehat{f_k}(\eta)\, dk\Big) = |\eta|^{(n-1)/2}\widehat{R_u f}(\eta).$$

Inserting this yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_M \langle f, G_{k,u}\rangle F_{k,u}(x)\,dk\,du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty} |\eta|^{(n-1)/2}\widehat{R_u f}(\eta)e^{2\pi i\eta u\cdot x}|\eta|^{(n-1)/2}\,d\eta\,du\\
&\quad = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty} |\eta|^{n-1}\widehat{R_u f}(\eta)e^{2\pi i\eta u\cdot x}\,d\eta\,du. \qquad (2.1)
\end{aligned}$$

From here on, standard arguments complete the proof. In fact, by the Fourier slice theorem, $\widehat{R_u f}(\eta) = \widehat{f}(\eta u)$; inserting this in (2.1), and splitting the integral over $\mathbb{R}$ into integrals over $]-\infty, 0]$ and $[0, \infty[$, a few changes of variables show that (2.1) equals $\int_{\mathbb{R}^n}\widehat{f}(y)e^{2\pi i x\cdot y}\,dy = f(x)$, as desired.

Case (ii) follows from the density of $\mathcal{S}(\mathbb{R})$ in $H^\alpha(\mathbb{R})$ for $\alpha \geq 0$, Lemma 1.2 and the fact that the Fourier transform is a continuous linear operator on $H^\alpha(\mathbb{R}^n)$ [2, Section 9.3]. □

We have already seen in Section 1 that it is easy to construct continuous tight wavelet frames for $L^2(\mathbb{R})$ that are generated by functions $\psi \in \mathcal{S}(\mathbb{R})$; thus, it is easy to give applications of Theorem 2.1. However, for the purpose of applications our ultimate goal is to provide discrete realisations of the theory, so we will not consider concrete cases here.

3. Semidiscrete representations

The integral representation in Theorem 2.1 involves integrals over the sphere $\mathbb{S}^{n-1}$ as well as the set M. Letting M be a discrete set equipped with the counting measure, we can of course apply the result to discrete frames as well; in this case we obtain what we will call a semidiscrete representation of functions $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, involving only an integral over $\mathbb{S}^{n-1}$ and a sum over the discrete index set. Nevertheless, we will follow the approach by Grafakos and Sensing, see [17, Theorem 5], where a semidiscrete representation is derived in the Gabor case, independently of the continuous case. The reason for doing this is that the technical conditions are slightly weaker in this approach, leading to a representation that is valid for a larger class of functions.

Theorem 3.1. *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let I be a countable index set.*

(i) *Let $\{g_k\}_{k\in I}$ denote a frame for $L^2(\mathbb{R})$ with frame bounds A, B and assume that either $\{f_k\}_{k\in M} \subset \mathcal{S}(\mathbb{R})$ or $\{f_k\}_{k\in M}$, $\{g_k\}_{k\in M} \subset H^\alpha(\mathbb{R})$, $\alpha > 0$. Define the associated functions G_k and $G_{k,u}$ as in Definition 1.1. Then*

$$2A\|f\|^2 \leq \int_{\mathbb{S}^{n-1}}\sum_{k\in I}|\langle f, G_{k,u}\rangle|^2\,du \leq 2B\|f\|^2.$$

(ii) *Assume that $\{g_k\}_{k\in I}$ and $\{f_k\}_{k\in I}$ are dual frames for $L^2(\mathbb{R})$ and that either $\{f_k\}_{k\in M}$, $\{g_k\}_{k\in M} \subset \mathcal{S}(\mathbb{R})$ or $\{f_k\}_{k\in M}$, $\{g_k\}_{k\in M} \subset H^\alpha(\mathbb{R})$, $\alpha > 0$. Then*

$$f = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\sum_{k\in I}\langle f, G_{k,u}\rangle F_{k,u}\,du.$$

Proof. We prove the result in the case of frames consisting of functions in $\mathcal{S}(\mathbb{R})$ and leave the obvious modifications in the Sobolev case to the reader. First, by Lemma 1.3, applied to the function G_k,

$$\langle f, G_{k,u}\rangle = \langle R_u f, G_k\rangle = \langle R_u f, \mathcal{D}^{(n-1)/2} g_k\rangle = \langle \mathcal{D}^{(n-1)/2} R_u f, g_k\rangle, \tag{3.1}$$

where the last equality follows by partial integration and the assumption $g_k \in \mathcal{S}(\mathbb{R})$. Now, by the frame assumption on $\{g_k\}_{k=1}^{\infty}$,

$$A\|\mathcal{D}^{(n-1)/2} R_u f\|^2 \le \sum_{k\in I} |\langle \mathcal{D}^{(n-1)/2} R_u f, g_k\rangle|^2 \le B\|\mathcal{D}^{(n-1)/2} R_u f\|^2. \tag{3.2}$$

As shown in [17],

$$\int_{\mathbb{S}^{n-1}} \|\mathcal{D}^{(n-1)/2} R_u f\|^2 \, du = 2\|f\|^2.$$

Thus, integrating (3.2) over $\mathbb{S}^{n-1}$ and applying (3.1) yields the result in (i).

As for the proof of (ii), the frame decomposition associated with the frames $\{f_k\}_{k\in I}$ and $\{g_k\}_{k\in I}$ and applied to the function $\mathcal{D}^{(n-1)/2} R_u f$ (which belongs to $L^2(\mathbb{R})$ by [17, Lemma 2]) yields

$$\begin{aligned} \mathcal{D}^{n-1} R_u f &= \mathcal{D}^{(n-1)/2}\mathcal{D}^{(n-1)/2} R_u f = \mathcal{D}^{(n-1)/2} \sum_{k\in I} \langle \mathcal{D}^{(n-1)/2} R_u f, g_k\rangle f_k \\ &= \sum_{k\in I} \langle \mathcal{D}^{(n-1)/2} R_u f, g_k\rangle \mathcal{D}^{(n-1)/2} f_k = \sum_{k\in I} \langle R_u f, G_k\rangle F_k = \sum_{k\in I} \langle f, G_{k,u}\rangle F_k. \end{aligned}$$

Since $f(x) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \mathcal{D}^{n-1} R_u(f)(u\cdot x)\, du$, it follows that

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \sum_{k\in I} \langle f, G_{k,u}\rangle F_k(u\cdot x)\, du = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \sum_{k\in I} \langle f, G_{k,u}\rangle F_k(u\cdot x)\, du \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \sum_{k\in I} \langle f, G_{k,u}\rangle F_{k,u}(x)\, du, \end{aligned}$$

as desired. □

It is easy to satisfy the assumptions in Theorem 3.1 (see, for example, [33, Theorem 3.4]). Let us illustrate the result by considering the Meyer wavelet.

Example 3.2. Let $\nu : \mathbb{R} \to [0,1]$ be a smooth function of sigmoidal shape required to satisfy $\nu(y) = 0$ for $y \le 0$, $\nu(y) = 1$ for $y \ge 1$ and $\nu(y) + \nu(1-y) = 1$. An example of such a function is the polynomial $\nu(y) = y^4(35 - 84y + 70y^2 - 20y^3)$ for $y \in [0,1]$.

Now let

$$w(y) := \begin{cases} \sin\Big(\dfrac{\pi}{2}\nu\Big(\dfrac{3y}{2\pi} - 1\Big)\Big) & \text{for } \dfrac{2\pi}{3} \le y \le \dfrac{4\pi}{3}, \\ \cos\Big(\dfrac{\pi}{2}\nu\Big(\dfrac{3y}{2\pi} - 1\Big)\Big) & \text{for } \dfrac{4\pi}{3} \le y \le \dfrac{8\pi}{3}, \\ 0 & \text{elsewhere.} \end{cases}$$

The classical Meyer wavelet ψ is defined in the Fourier domain by

$$\widehat{\psi}(\gamma) := e^{-i\pi\gamma}(w(2\pi\gamma) + w(-2\pi\gamma)).$$

It is well known that ψ is a Schwartz function and that

$$\{\psi_{k,m}\}_{k,m\in\mathbb{Z}} := \{2^{-m/2}\psi(2^{-m}\cdot - k) \mid k, m \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$ (see [12, 25, 33]). In particular, $\{\psi_{k,m}\}_{k,m\in\mathbb{Z}}$ is a frame, which is its own dual. Thus, we can apply Theorem 3.1; the functions $G_{k,u} = F_{k,u}$ have the form

$$\begin{aligned}\Psi_{k,m,u}(x) := \Psi_{k,m}(u \cdot x) &= \mathcal{D}^{(n-1)/2}\psi_{k,m}(u \cdot x)\\ &= (|\cdot|^{(n-1)/2}\widehat{\psi_{k,m}})^{\vee}(u \cdot x), \quad k, m \in \mathbb{Z}, u \in \mathbb{S}^{n-1}.\end{aligned}$$

Example 3.3. As a second example, we consider complex B-splines $\beta_z : \mathbb{R} \to \mathbb{C}$, which are a natural extension of the classical Curry–Schoenberg B-splines [15]. For a given $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, complex B-splines β_z are defined in the Fourier domain by

$$\widehat{\beta_z}(\gamma) := \left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^z =: \Omega(z)^z.$$

As graph $\Omega \cap \{(0, y) \in \mathbb{R} \times \mathbb{R} \mid y < 0\} = \emptyset$, complex B-splines reside on the main branch of the complex logarithm and are thus well defined.

Compared with the classical cardinal B-splines, complex B-splines β_z possess an additional modulation and phase factor in the frequency domain:

$$\widehat{\beta_z}(\gamma) = \widehat{\beta}_{\operatorname{Re} z}(\gamma)e^{i\operatorname{Im} z \ln|\Omega(\gamma)|}e^{-\operatorname{Im} z \arg \Omega(\gamma)}.$$

For more details of how these functions may be employed in image and signal analysis, we refer the interested reader to [14].

It follows from the time-domain representation of complex B-splines,

$$\beta_z(x) = \frac{1}{\Gamma(z)}\sum_{k=0}^{\infty}(-1)^k\binom{z}{k}(x-k)_+^{z-1},$$

that they cannot be elements of $\mathcal{S}(\mathbb{R})$. However, they belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and, for $0 < \alpha < \operatorname{Re} z - \frac{1}{2}$, to the Sobolev spaces $H^\alpha(\mathbb{R})$.

It was shown in [15] that complex B-splines generate a multiresolution analysis $\{V_k \mid k \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$. In particular, $\{\beta_z(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$ is a Riesz basis for V_0. Using the standard construction procedures for orthogonal scaling functions and orthogonal wavelets, we obtain by Theorem 3.1 the associated ridge wavelets. See Figures 1 and 2 for an illustrative example.

As already mentioned, the fact that Theorems 2.1 and 3.1 hold for frames in Sobolev spaces significantly extends the applicability. The list of explicitly known frames (or pairs of dual frames) includes the following ones.

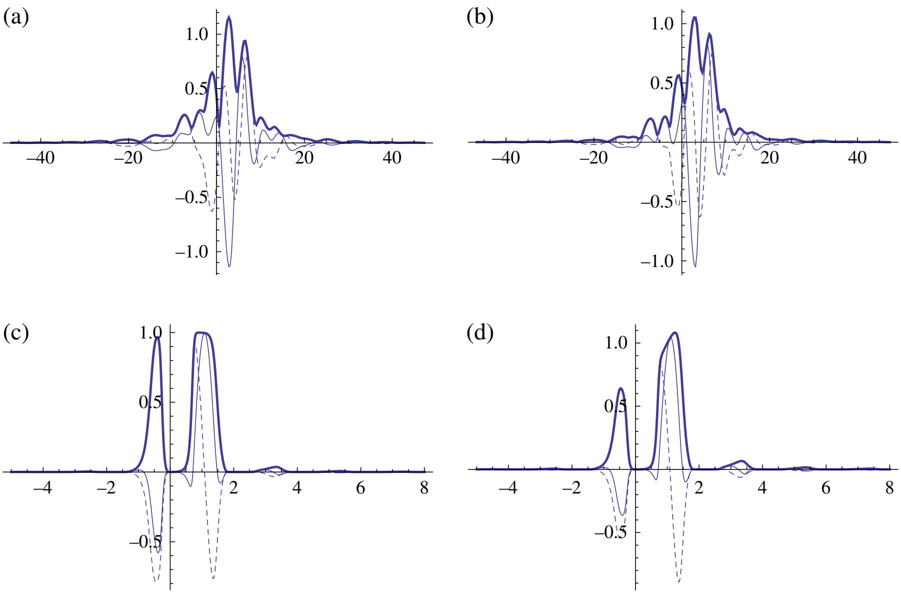

FIGURE 1. For $z = 3.5 + i$, the figure shows the time-domain representation of (a) the orthonormalised complex B-spline wavelet ψ; (b) its ridge variant Ψ; (c) the spectrum of ψ; (d) the spectrum of Ψ. Thick lines indicate the modulus, thin lines the real part and dashed lines the imaginary part.

- In the Gabor case, it is known how to construct frames generated by B-splines [8] or exponential splines [9], with a dual Gabor frame generated by a function that is a finite linear combination of shifted versions of the same B-spline; such frame generators belong to H^α for $\alpha < N - \frac{1}{2}$, where N denotes the order of the B-spline. It is also known how to construct dual pairs of frames based on continuous splines with compact support (see, for example, the paper by Laugesen [24] or the paper by Kim [22]).
- The unitary extension principle by Ron and Shen [29] and its more recent variants [10, 13] yield dual pairs of wavelet frames (or tight frames) generated by B-splines.

4. Discrete representations

In this section, we consider the cube,

$$Q := [-1, 1]^n \subset \mathbb{R}^n,$$

and functions $f \in L^2(Q)$. We will present a discretisation of the sphere $\mathbb{S}^{n-1}$, which ultimately leads to a complete discrete representation of functions $f \in L^2(Q)$. This discretisation was also considered in [4] and is based on the concept of an ε-net. It is one of several existing discretisation methodologies. (Other choices include the methods in [3, 11, 26].) Let us recall the definition of a finite ε-net.

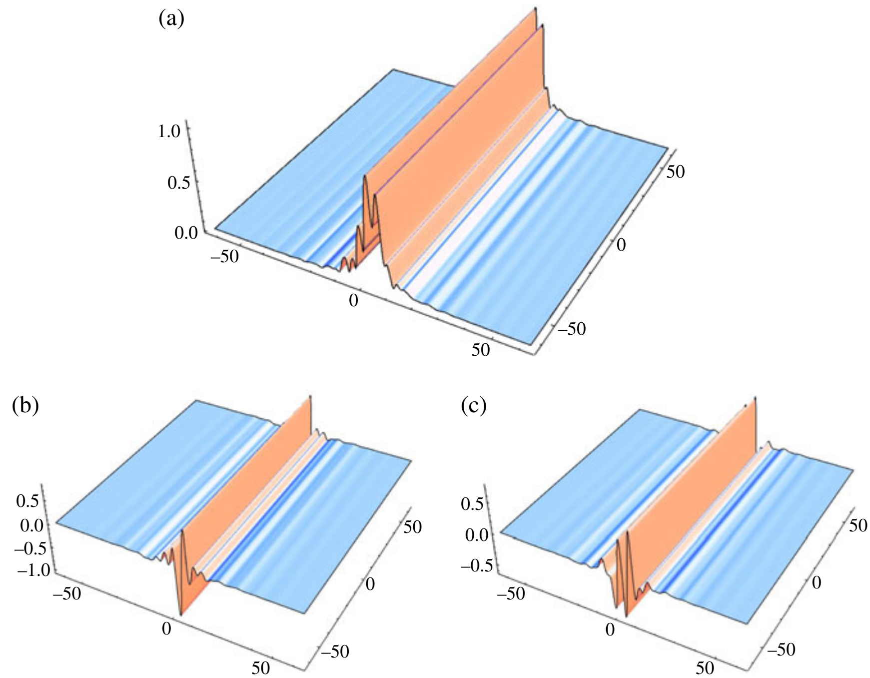

FIGURE 2. The modulus of the orthonormal spline ridge wavelet Ψ in the direction of $u = (1\ \ 0)^T$ for $z = 3.5 + i$.

DEFINITION 4.1. Let (X, d) be a metric space and take a discrete set $N \subset X$. Given any $\epsilon > 0$, the set N is called an ε-net for M if:

(a) $\inf\{d(y, y') \mid y \neq y', y, y' \in N\} \geq \varepsilon$;
(b) $\inf\{r \mid X = \bigcup_{y \in N} \overline{B}_r(y)\} \leq \varepsilon$, where $\overline{B}_r(y)$ denotes the closed ball of radius $r > 0$ centered at y.

An ε-net is called finite if N is a finite set.

Note that since the sphere is compact and hence totally bounded, an ε-net exists for $\mathbb{S}^{n-1}$ for all $\varepsilon > 0$ (see [31]).

We employ the following discretisation procedure for $\mathbb{S}^{n-1}$; see also [4].

(i) Choose an $a_0 > 1$ and discretise the scale parameter a by the sequence $\{a_k := a_0^k \mid k \in I\}$, where $I := \{k \in \mathbb{Z} \mid k \geq k_0\}$ and $k_0 \in \mathbb{Z}$ is selected appropriately.
(ii) For $k \in I$, set $\varepsilon_k := \frac{1}{2} a_{k_0 - k}$.
(iii) Let S_k^{n-1} be an ε_k-net of $\mathbb{S}^{n-1}$ and require that following condition holds: there exist positive constants $c = c(n)$ and $C = C(n)$ so that for all $u \in \mathbb{S}^{n-1}$ and all $\varepsilon_k \leq r \leq 2$,

$$c\left(\frac{r}{\varepsilon_k}\right)^{n-1} \leq \operatorname{card}\left(\{B_r(u) \cap S_k^{n-1}\}\right) \leq C\left(\frac{r}{\varepsilon_k}\right)^{n-1}.$$

Note that for $0 < r \leq \varepsilon_k$, $B_r(u) \subseteq B_{\varepsilon_k}(u)$ and thus $\mathrm{card}\,\{B_r(u) \cap S_k^{n-1}\} \leq C$. It can be proved that the number of points N_k in the ε_k-net satisfies the following bounds:

$$c\Big(\frac{r}{\varepsilon_k}\Big)^{n-1} \leq N_k \leq C\Big(\frac{r}{\varepsilon_k}\Big)^{n-1}.$$

Next, we list the standing assumptions for this section.

4.1. General setup. Let $g \in \mathcal{S}(\mathbb{R})$ and assume that:

(i) $\int_{-\infty}^{\infty} ((|\widehat{g}(\gamma)|^2)/(|\gamma|^n))\, d\gamma < \infty$;
(ii) $\inf_{1\leq|\gamma|\leq a_0} \sum_{k=0}^{\infty} |\widehat{g}(a_0^{-k}\gamma)|^2 |a_0^{-k}\gamma|^{-2(n-1)} > 0$;
(iii) $|\widehat{g}(\gamma)| \leq K|\gamma|^{\alpha}(1+|\gamma|)^{-\beta}$ for some $K > 0$, $\alpha > (n-1)/2$ and $\beta > \alpha + (n+3)/2$.

These conditions are satisfied for a large class of functions, for instance the Gaussian.

In particular, we remark that:

- if condition (i) is satisfied, then $G := \mathcal{D}^{(n-1)/2}\, g$ satisfies the admissibility condition (1.3);
- condition (ii) is satisfied if

$$\inf_{1\leq|\gamma|\leq a_0} \sum_{k=0}^{\infty} |\widehat{g}(a_0^{-k}\gamma)|^2 |a_0^{-k}\gamma|^{-(n-1)} > 0.$$

For the proof of our result we need [4, Theorem 4], which we state here in our notation.

THEOREM 4.2 [4, Theorem 4]. *Assume that the function $g \in C^1(\mathbb{R})$ satisfies the following two conditions:*

- $\inf_{1\leq|\gamma|\leq a_0} \sum_{k=0}^{\infty} |\widehat{g}(a_0^{-k}\gamma)|^2 |a_0^{-k}\gamma|^{-(n-1)} > 0$;
- $|\widehat{g}(\gamma)| \leq K|\gamma|^{\alpha}(1+|\gamma|)^{-\beta}$ *for some* $K > 0$, $\alpha > (n-1)/2$ *and* $\beta > 2 + \alpha$.

Then there exists $b_0 > 0$ so that for any $b < b_0$, we can find two constants $A, B > 0$ (depending on g, a_0, b_0 and n) so that, for any $f \in L^2(Q)$,

$$A\|f\|^2_{L^2(Q)} \leq \sum_{k\in I} \sum_{u\in S_k^{n-1}} \sum_{\ell\in\mathbb{Z}} |\langle f, D_{a_k} T_{\ell b} G_u\rangle|^2 \leq B\|f\|^2_{L^2(Q)}. \tag{4.1}$$

We will now show that under the general setup and with the discretisation of the unit sphere $\mathbb{S}^{n-1}$ in term of the ε-net introduced above, there exists a discrete frame for $L^2(Q)$.

THEOREM 4.3. *Let $g \in \mathcal{S}(\mathbb{R})$ be as in the general setup and let $G := \mathcal{D}^{(n-1)/2} g$. Then there exists a $b_0 > 0$ so that (4.1) holds for any given $b \in]0, b_0]$, that is, the orthogonal projection of $\{D_{a_k} T_{\ell b} G_u \mid k \in I; \ell \in \mathbb{Z}; u \in S_k^{n-1}\}$ onto $L^2(Q)$ forms a frame for $L^2(Q)$.*

Proof. Let G be defined as in (1.1). Then

$$|\widehat{G}(a_0^{-k}\gamma)|^2|a_0^{-k}\gamma|^{-2(n-1)} = |\widehat{g}(a_0^{-k}\gamma)|^2|a_0^{-k}\gamma|^{n-1}|a_0^{-k}\gamma|^{-2(n-1)} = |\widehat{g}(a_0^{-k}\gamma)|^2|a_0^{-k}\gamma|^{-(n-1)}$$

and, therefore,

$$\inf_{1\le|\gamma|\le a_0}\sum_{k=0}^{\infty}|\widehat{G}(a_0^{-k}\gamma)|^2|a_0^{-k}\gamma|^{-(n-1)} > 0.$$

Furthermore,

$$|\widehat{G}(\gamma)| = |\widehat{g}(\gamma)|\,|\gamma|^{(n-1)/2} \le K|\gamma|^{\alpha+(n-1)/2}(1+|\gamma|)^{-\beta}.$$

Hence, the function G satisfies the two conditions in Theorem 4.2 and the result follows. □

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