Orthogonality and asymptotics of Pseudo-Jacobi polynomials for non-classical parameters

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Abstract

The family of general Jacobi polynomials $P_n^{(\alpha,\beta)}$ where $\alpha,\ \beta \in \mathbb{C}$ can be characterised by complex (non-hermitian) orthogonality relations (cf. [15]). The special subclass of Jacobi polynomials $P_n^{(\alpha,\beta)}$ where $\alpha, \beta \in \mathbb{R}$ are classical and the real orthogonality, quasi-orthogonality as well as related properties, such as the behaviour of the n real zeros, have been well studied. There is another special subclass of Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha,\ \beta \in \mathbb{C}$, $\beta = \overline{\alpha}$ which are known as Pseudo-Jacobi polynomials. The sequence of Pseudo-Jacobi polynomials $\{P_n^{\alpha,\overline{\alpha}}\}_{n=0}^{\infty}$ is the only other subclass in the general Jacobi family (beside the classical Jacobi polynomials) that has n real zeros for every $n = 0, 1, 2, \dots$ for certain values of $\alpha \in \mathbb{C}$. For some parameter ranges Pseudo-Jacobi polynomials are fully orthogonal, for others there is only complex (non-Hermitian) orthogonality. We summarise the orthogonality and quasi-orthogonality properties and study the zeros of Pseudo-Jacobi polynomials, providing asymptotics, bounds and results on the monotonicity and convexity of the zeros.

Keywords: Orthogonal polynomials; quasi-orthogonal polynomials; Jacobi polynomials with complex parameters; Pseudo-Jacobi polynomials; zeros.
2000 MSC: 33C45, 42C05

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1. Introduction

The sequence of Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ defined by (cf. [14])

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!} \left(\frac{1-x}{2}\right)^k,\ n = 0, 1, 2 \dots$$

where $(a)_k = a(a+1)\dots(a+k-1)$ for $k = 1, 2, \dots$ and $(a)_0 = 1$ is the shifted factorial function, is orthogonal on $(-1, 1)$ when $\alpha,\ \beta > -1$ with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$. Jacobi polynomials arise in the study of rotation groups [26] and occur in the solution of Schrödinger's equation (cf. [4]). Furthermore, they play a role in approximation of functions for which the Laplace transform is known [1] and have many applications in areas other than mathematics such as, for example, in the medical field in ECG data compression [29].

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[1]Research of this author is supported by the National Research Foundation of South Africa

In [15] it was shown that Jacobi polynomials $P_n^{(\alpha,\beta)}$ can be characterised for general $\alpha,\ \beta \in \mathbb{C}$ by non-hermitian orthogonality relations. The corresponding weight is the multi-valued function

$$w(z) = w(z; \alpha, \beta) = (1-z)^{\alpha}(1+z)^{\beta} \tag{1}$$

and the orthogonality curve is defined on a Riemann surface, however, depending on α and β, the orthogonality curve can be transformed into one, two or three curves in the complex plane. Moreover, it was established in [15] that the orthogonality relations uniquely determine the Jacobi polynomials. The strong asymptotics and the asymptotic zero distribution for the case $\alpha,\ \beta \in \mathbb{R}$ have been investigated in [16] and [18] respectively.

In [12], a different complex weight function was obtained, which has the advantage that the curve of orthogonality is a simple closed curve independent of the value of α and β. This weight function is considered only for real α and β in [12], but applies to general $\alpha,\ \beta \in \mathbb{C}$, with the orthogonality

$$\frac{1}{2\pi i}\int_C P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)(z-1)}\ {}_2F_1\left(1, \alpha+1; \alpha+\beta+2; \frac{2}{1-z}\right) dz = 0, \quad m = 0, 1, ...n-1$$

for $|z-1| > 2$ defined in the same way as for $\alpha,\ \beta$ real parameter values (cf. [12, eqn. (2.7)]). Nevertheless, the form of this weight function is more complicated and less appropriate for the analysis of the zero location or asymptotics than the weight (1) given in [15].

An interesting subclass of Jacobi polynomials $P_n^{(\alpha,\beta)}$, $\alpha, \beta \in \mathbb{C}$, is the class where $\beta = \overline{\alpha} \notin \mathbb{R}$, known as Pseudo-Jacobi polynomials, which form a finite sequence of orthogonal polynomials (cf. [21], [2]) associated with a beta integral due to Cauchy (cf. [3, (1.9)]. They are defined by (cf. [10, p.509], [14])

$$\begin{aligned} P_n(x; a, b) &= (-i)^n P_n^{(a+ib, a-ib)}(ix) \\ &= (-i)^n \frac{(a+ib+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (2a+n+1)_k}{(a+ib+1)_k} \frac{(1-ix)^k}{k! 2^k}, \ n = 0, 1, 2, \ldots. \end{aligned}$$

Note that the normalisation given here is different from that in [14]. The differential equation

$$(x^2+1)y''(x) + 2[(a+1)x + b]y'(x) - n(n+2a+1)y(x) = 0, \tag{2}$$

satisfied by $y(x) = P_n(x; a, b)$, was already studied in 1884 by Routh (cf. [22]). It is easy to see from the three term recurrence relation (cf. [14])

$$\begin{aligned} &(n+1)(n+2a+1)(n+a)P_{n+1}(x; a, b) \\ &= (2n+2a+1)(x(n+a)(n+a+1)+ab)P_n(x; a, b) + ((n+a)^2+b^2)(n+a+1)P_{n-1}(x; a, b) \end{aligned} \tag{3}$$

with $P_{-1} = 0$ and $P_0 = 1$ that $P_n(x; a, b)$, $n = 1, 2, \ldots$, has real coefficients for $a,\ b \in \mathbb{R}$. Pseudo-Jacobi polynomials are unique in the sense that they are the only polynomials in the Jacobi class with non-real parameters and imaginary argument that have real coefficients. This is evident from the fact that $P_n^{(\alpha,\beta)}(ix)$, $n \in \mathbb{N}$, has real coefficients for all n even and purely imaginary coefficients for all odd n only when $\alpha = \bar{\beta}$. To see this, it suffices to consider the coefficients of $P_1^{(\alpha,\beta)}(ix)$ and $P_2^{(\alpha,\beta)}(ix)$. In fact, it is possible to show, using a symbolic language such as Mathematica, that a more general polynomial of the form $(e+if)^n P_n^{(a+ib,c+id)}((g+ih)x)$ with $a, b, c, d, e, f, g, h \in \mathbb{R}$, has real coefficients for all n only when $b = d = f = h = 0$ (the classical Jacobi case) or $b = -d$, $c = a$, $e = 0$ and $g = 0$ (the Pseudo-Jacobi case) and to see this it is enough to check up to degree 4.

The symmetry property

$$P_n(x; a, -b) = (-1)^n P_n(-x; a, b), \tag{4}$$

which follows immediately from the symmetry property (cf. [10, p.82, eqn. (4.1.1)])

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

for Jacobi polynomials, implies that, when the sign of b changes, the zeros of Pseudo-Jacobi polynomials reflect around the imaginary axis for $n \in \mathbb{N}$, $a \in \mathbb{R}$.

With the substitution $\alpha = a + ib$, $\beta = a - ib$ and $z = ix$, the general weight function (1) gives rise to the real weight function for Pseudo-Jacobi polynomials

$$w_0(x) := (1 + x^2)^a e^{2b \arctan x}$$

whose moments exist up to $n = \lfloor -2a \rfloor - 1$ (cf. [3, (1.10)]) where $\lfloor x \rfloor$ denotes the lower integer part of x. This provides (real) orthogonality relations on the real line for $a < -\frac{n}{2} - \frac{1}{2}$ which we discuss in section 3. Note that, since not all the moments exist, only a finite number of polynomials will be orthogonal. From this orthogonality one has to exclude the special cases when $n + 2a = -1, -2, \ldots, -n$, i.e. $a = -\frac{n}{2} - \frac{1}{2}, -\frac{n}{2} - 1, \ldots, -n$, where the degree of $P_n(x; a, b)$ becomes less than n and $z = \infty$ becomes a multiple zero. When $a < -n$ we obtain full orthogonality on the real line (at least up to degree n) and this case is considered in more detail in the next section. The remaining cases, when $-\frac{n}{2} - \frac{1}{2} < a < -1$ and $a > -1$, are discussed in Sections 4 and 5 respectively.

2. $P_n(x; a, b)$, $a < -n$

When $a < -n$, Pseudo-Jacobi polynomials satisfy the finite orthogonality relation (cf. [2, 3, 14, 24])

$$\int_{-\infty}^{\infty} w_0(x) P_n(x; a, b) P_m(x; a, b) dx = 0 \quad \text{for} \quad m = 1, 2, \ldots, n-1.$$

Hence, when $a < -n$, the zeros of $P_n(x; a, b)$ are real and simple.

Since, by setting $b = 0$, the polynomial $P_n(x; a, 0)$ is a rotated version of the Gegenbauer polynomials $C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x)$, we have the following corollary.

Lemma 2.1. *(cf. [2, Theorem 1.1]) If $\lambda < -n + 1/2$ then all the zeros of the Gegenbauer polynomial $C_n^{(\lambda)}(x)$ are simple and purely imaginary.*

2.1. Asymptotic zero distribution

To consider the asymptotic distribution of the zeros as $n \to \infty$, we need an appropriate scaling and therefore we assume that a and b vary linearly with n. More precisely, we assume that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences such that

$$\lim_{n\to\infty} \frac{a_n}{n} = A \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = B, \tag{5}$$

where $A, B \in \mathbb{R}$. We are interested in the limit of the normalised zero-counting measure ν_n of $P_n(x; a_n, b_n)$. We can state the following.

Theorem 2.2. *Let the sequence of Pseudo-Jacobi polynomials $\{P_n(x; a_n, b_n)\}_{n=0}^{\infty}$ satisfy (5) and assume that $A < -1$. Then there is a unique unit measure ν, which is the weak-*-limit of ν_n. This ν is supported on $[\gamma_-, \gamma_+]$, where*

$$\gamma_{\pm} = \frac{-AB \pm \sqrt{-2(A + 1/2)[(A+1)^2 + B^2]}}{(A+1)^2}. \tag{6}$$

Moreover, ν is absolutely continuous with respect to the Lebesgue measure with density

$$\frac{d\nu}{dt} = \frac{-(A+1)\sqrt{(t-\gamma_-)(\gamma_+-t)}}{\pi(t^2+1)}. \tag{7}$$

Proof. The proof follows a standard method due to Perron (cf. [19, 23] and also [18]). First we have to establish that the zeros are uniformly bounded. This can be done using the monic three term recurrence relation and Gershgorin's circle theorem. The monic three term recurrence relation is

$$xp_n(x;a,b) = p_{n+1}(x;a,b) + c_n p_n(x;a,b) + d_n p_{n-1}(x;a,b),$$

where $p_n(x;a,b)$ denotes the monic Pseudo-Jacobi polynomial,

$$c_n = -\frac{ab}{(n+a)(n+a+1)} \quad \text{and} \quad d_n = -\frac{n(n+2a)[(n+a)^2+b^2]}{(n+a)^2(2n+2a-1)(2n+2a+1)}.$$

With $a = a_n$, $b = b_n$ and using (5) we have

$$\lim_{n\to\infty} c_n = -\frac{AB}{(A+1)^2} \quad \text{and} \quad \lim_{n\to\infty} d_n = -\frac{(2A+1)[(A+1)^2+B^2]}{4(A+1)^4}.$$

Thus, it follows from Gershgorin's theorem (cf. [20]) that, for $A \neq -1$, the zeros are uniformly bounded. It means that the sequence $\{\nu_n\}_{n=0}^{\infty}$ is weakly compact. Thus, there is a subsequence which weak-*-converges to a unit measure ν. The uniqueness of this measure follows from the fact that it can be explicitly given using the Stieltjes transform.

Let h_n be the Stieltjes transform of ν_n, i.e

$$h_n(z) := \int \frac{d\nu_n(t)}{t-z} = -\frac{p_n'(z)}{np_n(z)}.$$

From (2) we can obtain a differential equation for h_n

$$(z^2+1)\left(h_n^2(z) - \frac{h_n'(z)}{n}\right) - \frac{2[(a+1)z+b]}{n}h_n(z) - \frac{n+2a+1}{n} = 0. \tag{8}$$

Since the zeros are uniformly bounded, the $h_n(z)$'s are also uniformly bounded if z is in a domain away from the zeros. Thus by Montel's theorem there is a subsequence which converges locally uniformly on this domain to the Stieltjes transform $h(z)$ of ν

$$h(z) := \int \frac{d\nu(t)}{t-z}.$$

Now, taking $n \to \infty$ in (8) we get the algebraic equation

$$(z^2+1)h^2(z) - 2(Az+B)h(z) - (2A+1) = 0$$

for h, which leads to

$$h(z) = \frac{Az+B+\sqrt{Q(z)}}{z^2+1}, \tag{9}$$

where

$$Q(z) = (A+1)^2z^2 + 2ABz + 2A + B^2 + 1 = (A+1)^2(z-\gamma_-)(z-\gamma_+),$$

$\gamma_\pm$ being defined by (6). Applying the Stieltjes-Perron inversion formula to h (see e.g. [30, formula (65.4)]), we obtain the statement of the theorem. ∎

Figure 1 shows the zeros and the asymptotic distribution according to Theorem 2.2.

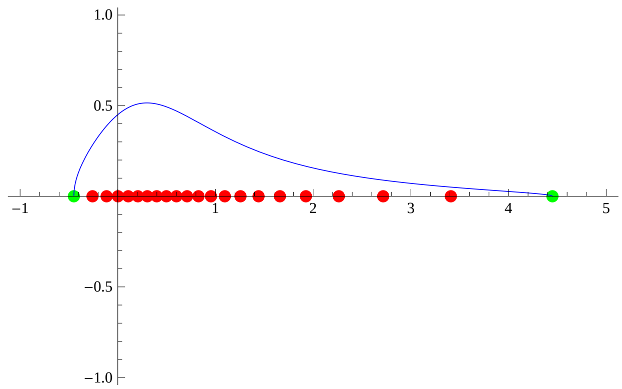

Figure 1: The zeros of $P_{20}(x;-40,20)$ together with the corresponding limiting distribution (7) and endpoints (6) with $A=-2$ and $B=1$

2.2. Bounds for the extreme zeros

We begin by proving some new lower bounds for the largest zeros and upper bounds for the smallest zeros of $P_n(x;a,b)$.

Theorem 2.3. *Fix $n\in\mathbb{N}$ and let $a<-n$. We denote the zeros of $P_n(x;a,b)$ by $x_{n,1}<x_{n,2}<\cdots<x_{n,n}$, then*

(i) $x_{n,1}<e_n=\dfrac{-ab}{(n+a)(n+a+1)(2n+2a+1)}<x_{n,n}$

(ii) $x_{n,1}<f_n=\dfrac{-b}{a+n}<x_{n,n}$

(iii) $x_{n,1}<g_n=\dfrac{-b}{a+1}<x_{n,n}$

Proof.

(i) This follows from the three term recurrence relation (3) by using Corollary 2.2 in [8].

(ii) Replacing n by $n-1$, α by $a+ib$, β by $a-ib$ and x by ix in the recurrence relation [7, eqn. (18)]) involving Jacobi polynomials $P_{n-2}^{(\alpha+1,\beta)}$, $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$, we obtain after some simplification,

$$-\frac{1+x^2}{2}P_{n-2}(x;a+1,b)=(x+\frac{b}{n+a})P_{n-1}(x;a,b)-\frac{n}{n+a}P_n(x;a,b).$$

The result then follows from [8, Corollary 2.2].

(iii) The relation

$$B_n(1-x^2)^2P_{n-2}^{(\alpha+2,\beta+2)}(x) \quad = \quad \left(x-\frac{\beta-\alpha}{\alpha+\beta+2}\right)P_{n-1}^{(\alpha,\beta)}(x)-A(x)P_n^{(\alpha,\beta)}(x) \tag{10}$$

where

$$B_n \quad = \quad \frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)(2n+\alpha+\beta)}{8(n+\alpha)(n+\beta)(\alpha+\beta+2)} \text{ and}$$

$$A(x) \quad = \quad \frac{n(x^2(n-1)(2n+\alpha+\beta)+2x(n-1)(\beta-\alpha)-2n-2n^2-\alpha-3n(\alpha+\beta)-\beta-4\alpha\beta}{4(n+\alpha)(n+\beta)(\alpha+\beta+2)}$$

follows from [7, (5), (12)] and can be verified by comparing coefficients on both sides of the equation. The substitutions α by $a+ib$, β by $a-ib$ and x by ix in (10) give

$$\begin{aligned}
&\frac{-n(n^2+3an+n+2b^2+2a^2+a-2b(n-1)x+(n-1)(n+a)x^2)}{2(a+1)((n+a)^2+b^2)}P_n(x;a,b)\\
&= \frac{(n+a)(n+2a+1)(n+2a+2)(1+x^2)^2}{8(a+1)((n+a)^2+b^2)}P_{n-2}(x;a+2,b)+(x+\frac{b}{a+1})P_{n-1}(x;a,b)
\end{aligned}$$

and [8, Corollary 2.2] yields the stated result.

■

Remark 1. *(i) As mentioned in the introduction, for fixed n, if $a=-n$ then $P_n(x;a,b)$ is only of degree $n-1$. Therefore we can expect that if we fix n and let $a\to -n^-$, one of the zeros of $P_n(x;a,b)$ will tend to either ∞ or $-\infty$. Indeed, this can be confirmed for $b>0$ using Theorem 2.3 (ii). Since $\lim_{a\to -n^-} f_n=\infty$, it follows that the largest zero of $P_n(x;a,b)$, which is bounded below by f_n, tends to ∞ as $a\to -n^-$. Keeping in mind the symmetry (4) this implies that for $b<0$, the smallest zero of $P_n(x;a,b)$ tends to $-\infty$ as $a\to -n^-$.*

(ii) When $b\geq 0$, we note that $f_n\geq g_n\geq 0$ and therefore f_n provides a better lower bound for the largest zero. If we also assume that $a<-n-1/2$, it follows that $e_n<0$ and hence, in this case, e_n will be the better upper bound for the smallest zero.

Next we consider a method based on the Wall-Wetzel Theorem, introduced by Ismail and Li in [11] and reproduced in [10], to obtain bounds for the interval on which the zeros lie.

Theorem 2.4. *Let $n\in\mathbb{N}$ fixed, $a<-n$ and $b>0$. Denote the zeros of $P_n(x;a,b)$ by $x_{n,1}<x_{n,2}<\cdots<x_{n,n}$, then*

(i) $x_{n,n}<x_n=\frac{-ab}{(n+a-1)^2-1}-\frac{1}{n+a-1}\sqrt{\frac{a^2b^2}{((n+a-1)^2-1)^2}-\frac{4(n-1)(n+2a-1))((n+a-1)^2+b^2)}{4(n+a-1)^2-1}\cos^2\frac{\pi}{n+1}}$

(ii) $x_{n,1}>y_n=\min_{1\leq k\leq n-1}\left\{\frac{-ab}{(k+a)^2-1}+\frac{1}{k+a}\sqrt{\frac{a^2b^2}{((k+a)^2-1)^2}-\frac{4k(k+2a)((k+a)^2+b^2)}{4(k+a)^2-1}\cos^2\frac{\pi}{n+1}}\right\}$.

Proof. Applying [11, Theorem 2 and 3] to the three term recurrence relation (3) by setting $\alpha_n=\frac{(n+1)(n+2a+1)}{(2n+2a+1)(n+a+1)}$, $\beta_n=\frac{-ab}{(n+a)(n+a+1)}$ and $\gamma_n=-\frac{(n+a)^2+b^2}{(2n+2a+1)(n+a)}$, we obtain

$$x_n=\max_{1\leq k\leq n-1}\left\{\frac{-ab}{(k+a)^2-1}-\frac{1}{k+a}\sqrt{\frac{a^2b^2}{((k+a)^2-1)^2}-\frac{4k(k+2a)((k+a)^2+b^2)}{4(k+a)^2-1}\cos^2\frac{\pi}{n+1}}\right\}$$

and (ii). Now (i) follows from the fact that the functions $\frac{1}{(k+a)^2-1}$, $-k(k+2a)$ and $\frac{(k+a)^2+b^2}{(k+a)^2}$ are positive and monotone increasing in k on $1,\ldots,n-1$.

■

Remark 2. *It is interesting to consider the asymptotic limit of x_n, with $a=An$ and $b=Bn$. In fact, we have that*

$$\lim_{n\to\infty}x_n \quad = \quad \lim_{n\to\infty}\left(\frac{-ABn^2}{(n+An-1)^2-1}-\frac{1}{n+An-1}\sqrt{\frac{A^2B^2n^4}{((n+An-1)^2-1)^2}-\frac{4(n-1)(n+2An-1))((n+An-1)^2+B^2n^2)}{4(n+An-1)^2-1}\cos^2\frac{\pi}{n+1}}\right)$$

$$= \quad \frac{-AB + \sqrt{-(2A+1)((A+1)^2+B^2)}}{(A+1)^2} = \gamma_+,$$

where γ_+ *is the upper bound of the support of the weak-*-limit of the normalised zero-counting measure* ν_n *defined in (6). Thus, the bound for the largest zero is asymptotically accurate.*

Corollary 2.5. *Let* $n \in \mathbb{N}$ *fixed,* $a < -n - 1/2$ *and* $b = 0$*. Denote the zeros of* $P_n(x; a, b)$ *by* $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$*, then*

(i) $x_{n,n} < x_n = \sqrt{\frac{-4(n-1)(n+2a-1)}{4(n+a-1)^2-1}\cos^2\frac{\pi}{n+1}}$

(ii) $x_{n,1} > y_n = -\sqrt{\frac{-4(n-1)(n+2a-1)}{4(n+a-1)^2-1}\cos^2\frac{\pi}{n+1}}$.

Proof. If we let $b = 0$ in Theorem 2.4(ii), we see that the expression in the curly brackets is negative and monotone decreasing in k for $k = 1, 2, \ldots, n-1$ and therefore attains its minimum for $k = n-1$.

■

2.3. Spacing of the zeros

From the differential equation (2) we can also obtain information on the distance between consecutive zeros. Let $x_1 < x_2 < \ldots < x_n$ denote the zeros of $P_n(x; a, b)$. We say that the zeros are convex if the distance between the zeros increases from left to right, i.e. $x_{k+2} - x_{k+1} > x_{k+1} - x_k$. If the distance decreases, i.e. $x_{k+2} - x_{k+1} < x_{k+1} - x_k$, the zeros are called concave. The convexity can be determined by Sturm's theorem from the so-called normal form of the differential equation (cf. [27, 6]). The normal form can be obtained by eliminating the first order term, which gives in our case

$$y''(x) + F(x)y(x) = 0,$$

where

$$F(x) = -\frac{1}{(x^2+1)^2}\left[(n+a)(n+a+1)x^2 + 2abx + (n^2 + 2an + n + a + b^2 + 1)\right].$$

According to Sturm's convexity theorem the zeros are concave on the interval where $F(x)$ is strictly increasing and convex where $F(x)$ is decreasing. Now,

$$F'(x) = 2\frac{j(x)}{(x^2+1)^3},$$

where $j(x) = (n+a)(n+a+1)x^3 + 3abx^2 + (n^2 + 2an + n - a^2 + a + 2b^2 + 2)x - ab$. Therefore the zeros are concave where the cubic polynomial $j(x) > 0$ and convex where $j(x) < 0$. Unfortunately the zeros of $j(t)$ are difficult to handle in general. However, one can obtain the convexity intervals for the Gegenbauer case. Setting $b = 0$ the zeros of $j(x)$ are at $x = 0$ and

$$x_{1,2} = \pm\sqrt{\frac{-n^2 - 2an - n + a^2 - a - 2}{(n+a)(n+a+1)}}.$$

We can compare these "inflection" points with the bounds for the zeros in Corollary 2.5.

$$x_n = \sqrt{\frac{-4(n-1)(n+2a-1)}{4(n+a-1)^2-1}\cos^2\frac{\pi}{n+1}} < \sqrt{\frac{-(n-1)(n+2a-1)}{(n+a-1)^2-1/4}} < \sqrt{\frac{-n^2 - 2an - n + a^2 - a - 2}{(n+a)(n+a+1)}} = x_2,$$

where the last inequality follows from the direct comparison of the numerators and denominators. Thus, we obtain the following result.

Theorem 2.6. *The convexity of the zeros of $P_n(x;a,0)$ changes only at the origin. The negative zeros are concave, the positive ones are convex.*

2.4. Monotonicity of the zeros

Markov's theorem (cf. [28, Theorem 6.12.1]) is a useful tool for obtaining monotonicity properties of zeros and yields the following result for the monotonicity of the zeros of $P_n(x;a,b)$ with respect to changes in the parameter b.

Theorem 2.7. *For $a<-n$ the zeros of $P_n(x;a,b)$ increase when the parameter b increases.*

Proof. The ratio $\frac{\frac{d}{db}(w_0(x))}{w_0(x)} = 2\arctan x$ which is an increasing function of x and therefore the result follows immediately from Markov's theorem (cf. [28, Theorem 6.12.1]. ■

Markov's theorem provides no information on the monotonicity of zeros of Pseudo-Jacobi polynomials with respect to changes in the parameter a since $\frac{\frac{d}{da}(w_0(x))}{w_0(x)} = \ln(1+x^2)$ is not a monotone function. However, an extension of Markov's Theorem to zeros of polynomials orthogonal with respect to an even weight (cf. [13]) does provide information on the monotonicity of zeros of Pseudo-Jacobi polynomials for the special case where $b=0$.

Theorem 2.8. *Let $a<-n$ and $b=0$, then the positive zeros of $P_n(x;a,0)$ increase when the parameter a increases.*

Proof. When $b=0$, the weight function $w_1(x)=(1+x^2)^a$ is an even function of x and hence the zeros of $P_n(x;a,0)$ will be symmetric about the origin. Since $\frac{\frac{d}{da}(w_1(x))}{w_1(x)} = \ln(1+x^2)$ is an increasing function of x for $x>0$ the result follows by [13, Theorem 3]. ■

3. $P_n(x;a,b)$, $-n<a<-\frac{n}{2}-\frac{1}{2}$

In this case, at the values $a=-n/2-1/2,\ -n/2-1,\ \ldots,\ -n$ there is a degree reduction in $P_n(x;a,b)$. If we fix n and increase a from $-n$, the zeros will gradually change from real to non-real as we pass these values. Of course, the polynomial having real coefficients, the complex zeros always appear in conjugate pairs.

The general complex orthogonality relation with respect to $w(z)$ can be broken down into orthogonality relations along 3 different curves like in [15, Section 6]. Here we state this result without proof. Let Γ_i be the circle $\{z\in\mathbb{C}:|z+i|=2\}$ and Γ_{-i} be the circle $\{z\in\mathbb{C}:|z-i|=2\}$, both oriented clockwise.

Proposition 3.1. *(cf. [15, Theorem 6.4]) Assume that $-n<a<-\frac{n}{2}-\frac{1}{2}$ and $a\neq -n+1/2,\ -n+1,\ \ldots,\ -n-1$. Then*

$$\int_{\Gamma_i} t^k P_n(t;a,b)w(it;a+ib,a+\lfloor -a\rfloor-ib)dt=0,\quad k=0,\ 1,\ \ldots,\ n-\lfloor -a\rfloor-1,$$

$$\int_{\Gamma_{-i}} t^k P_n(t;a,b)w(it;a+\lfloor -a\rfloor+ib,a-ib)dt=0,\quad k=0,\ 1,\ \ldots,\ n-\lfloor -a\rfloor-1,$$

and

$$\int_{-\infty}^{\infty} x^k P_n(x;a,b)w_0(x)dx=0,\quad k=0,\ 1,\ \ldots,\ 2\lfloor -a\rfloor-n-1. \tag{11}$$

This gives n orthogonality conditions together. Relations (11) are real orthogonality relations, but, unlike in Section 2, they only provide quasi-orthogonality. A polynomial P_n of exact degree $n \geq r$, is quasi-orthogonal of order r on $[a, b]$ with respect to a weight function $w(x) > 0$, if (cf. [5, p.159])

$$\int_a^b x^j P_n(x) w(x) dx \begin{cases} = 0, & \text{for } j = 0, 1, \ldots, n-r-1 \\ \neq 0, & \text{for } j = n-r. \end{cases}$$

Since $P_k(x; a.b)$ and $P_l(x; a, b)$ are orthogonal to each other with respect to $w_0(x)$ if $k + l \leq \lfloor -2a \rfloor - 1$, it follows that $P_n(x; a, b)$ is quasi-orthogonal of order $2n - \lfloor -2a \rfloor$. This gives a lower bound on the number of real zeros in this case (cf. [5, Theorem 2]).

Corollary 3.2. *Assume that $-n < a < -\frac{n}{2} - \frac{1}{2}$ and $a \neq -n + 1/2,\ -n+1,\ \ldots,\ -n-1$. Then $P_n(x; a, b)$ has at least $\lfloor -2a \rfloor - n$ real zeros.*

We can state again the corresponding result for Gegenbauer polynomials.

Corollary 3.3. *Assume that $-n + 1/2 < \lambda < -\frac{n}{2}$ and $\lambda \neq -n+1,\ -n+3/2,\ \ldots,\ -n-1/2$. Then $C_n^{(\lambda)}(x)$ has at least $\lfloor -2\lambda \rfloor - n + 1$ purely imaginary zeros.*

The asymptotic behaviour of the zeros is quite complicated in this case due to the fact that we have non-Hermitian orthogonality and three different curves with orthogonality conditions. This type of problem could be tackled by a Riemann-Hilbert approach using steepest descent analysis as was done in [16].

4. $P_n(x; a, b)$, $-\frac{n}{2} - \frac{1}{2} < a < -1$

In this case there is no real (quasi-) orthogonality anymore, nevertheless, just like in Section 3 the general complex orthogonality can be expressed in the form of separate orthogonality conditions on 3 curves (cf. [15, Theorem 6.3]).

Proposition 4.1. *Assume that $-\frac{n}{2} - \frac{1}{2} < a < -1$. Then*

$$\int_{\Gamma_i} t^k P_n(t; a, b) w(it; a+ib, a+\lfloor -a \rfloor - ib) dt = 0, \quad k = 0,\ 1,\ \ldots,\ -\lfloor -a \rfloor - 1,$$

$$\int_{\Gamma_{-i}} t^k P_n(t; a, b) w(it; a+\lfloor -a \rfloor + ib, a - ib) dt = 0, \quad k = 0,\ 1,\ \ldots,\ -\lfloor -a \rfloor - 1,$$

$$\int_{-i}^{i} t^k P_n(t; a, b) w(it; a+\lfloor -a \rfloor + ib, a+\lfloor -a \rfloor - ib) dt = 0, \quad k = 0,\ 1,\ \ldots,\ n - 2\lfloor -a \rfloor - 1, \tag{12}$$

where the last integration is along the imaginary interval $[-i, i]$.

Note that in (12) the weight function is $w_0(x)$, thus it is real for real x, however the integration curve is complex, so it does not give a real orthogonality relation.

Remark 3. *In case $b = 0$, formula (12) gives the quasi-orthogonality conditions for the Gegenbauer polynomials (cf. [5, Section 3.1]).*

5. $P_n(x;a,b)$, $a > -1$

The general complex orthogonality can be given in this case as a full set of orthogonality conditions along one single complex contour, the interval $[-i,i]$ (cf. [15, Corollary 5.2]).

Proposition 5.1. *Assume that $a > -1$. Then*

$$\int_{-i}^{i} t^k P_n(t;a,b) w(it;a+ib,a-ib)dt \begin{cases} = 0, & \text{for } k = 0,\ 1,\ \ldots,\ n-1, \\ \neq 0, & \text{for } k = n. \end{cases}$$

These conditions characterise the polynomials $P_n(x;a,b)$ of degree n up to a constant factor.

Although we have full orthogonality on a single contour, it is non-Hermitian, therefore we cannot conclude that the zeros would lie on $[-i,i]$. Nevertheless, according to numerical examples, they seem to lie on well-arranged complex curves. This can be confirmed, at least in the asymptotic sense, if $n \to \infty$.

Again, like in Section 2, assume that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences satisfying (5), where now $A > 0$. We would like to find the limit of the normalised zero-counting measure ν_n of $P_n(x;a_n,b_n)$. Although we can obtain the Stieltjes transform (9) of the limiting measure the same way as in Theorem 2.2, this time we cannot use the Stieltjes-Perron inversion formula to get the measure. Instead, one has to use potential theory to obtain the limiting measure. The method was developed by Stahl in [25] and Gonchar and Rakhmanov in [9]. They connected the complex orthogonality with a logarithmic energy-minimising problem on the complex plane in the presence of an external field. The solution of the energy problem, the so-called equilibrium measure, is the limiting measure itself.

This method was then generalised by Martínez-Finkelshtein (cf. [17] and [18, Theorem 2]) to the case when the orthogonality curve includes the point infinity, which is a branch point of the weight function. In our case we would need a further generalisation of this theorem to the case when the weight function has 2 branch points on the curve (for Pseudo-Jacobi these would be -1 and 1). This would, however, go beyond the scope of this paper. Nevertheless, according to simulations with Mathematica, the following statement seems to hold, which we state as a conjecture.

Conjecture 5.2. *(cf. [18, Theorem 1])*

Let the sequence of Pseudo-Jacobi polynomials $\{P_n(x;a_n,b_n)\}_{n=0}^{\infty}$ satisfy (5) and assume that $A > 0$. Then there is a unique unit measure ν, which is the weak--limit of ν_n. This ν is supported on an analytic arc Γ with endpoints $\gamma_\pm$ defined by (6). Γ is given by the equation*

$$Re \int_{\gamma_-}^{z} \frac{\sqrt{(t-\gamma_-)(t-\gamma_+)}}{1+t^2} dt = 0,$$

and the density of ν with respect to the Lebesgue measure is

$$\frac{d\nu}{|dt|} = \frac{2(A+1)}{\pi} \left| \frac{\sqrt{(t-\gamma_-)(t-\gamma_+)}}{1+t^2} \right|.$$

Remark 4. *This conjecture seems to hold true for the case $-\frac{n}{2} - \frac{1}{2} < a < -1$ as well (see Section 4), although in that case the orthogonality condition is given on three different curves instead of a single one.*

Acknowledgement

The authors would like to thank Mourad E. H. Ismail for calling their attention to the quasi-orthogonality of the Pseudo-Jacobi polynomials. We are also grateful to the anonymous referee whose useful comments helped to improve the manuscript.

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