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Dimension hopping and families of strictly positive definite zonal basis functions on spheres

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Abstract

Positive definite functions of compact support are widely used for radial basis function approximation as well as for estimation of spatial processes in geostatistics. Several constructions of such functions for $\mathbb{R}^d$ are based upon recurrence operators. These map functions of such type in a given space dimension onto similar ones in a space of lower or higher dimension. We provide analogs of these dimension hopping operators for positive definite, and strictly positive definite, zonal functions on the sphere. These operators are then used to provide new families of strictly positive definite functions with local support on the sphere.

Keywords: Zonal functions; Positive definite; Dimension hopping operators

1. Introduction

This paper investigates certain dimension hopping operators on spheres that preserve strict and non-strict positive definiteness of zonal functions. The operators are the analogs for the sphere

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of the dimension hopping *montée* and *descente* operators of Matheron [13] for radial functions $f : \mathbb{R}^d \to \mathbb{R}$. These latter operators were later rediscovered by Schaback and Wu [15]. Using the *montée* operator for the sphere, and some known strictly positive definite, zonal functions, we construct further families of locally supported, strictly positive definite zonal functions. For the purposes of computation it is useful that these new functions can be evaluated at a relatively low computational cost rather than being given by infinite series. The first construction is an analog for the sphere of the Wendland family [19] of radial basis functions for $\mathbb{R}^d$ starting from the function $A(x) = (1 - \|x\|_2)_+^{\lceil \frac{d+1}{2} \rceil}$ of Askey [2], which is strictly positive definite on $\mathbb{R}^d$. The unit sphere in $\mathbb{R}^{d+1}$ will be denoted by $\mathbb{S}^d$. Later in the paper a relationship is established between convolutions of zonal functions on $\mathbb{S}^{d+2}$ and those for related zonal functions on $\mathbb{S}^d$. This enables the construction of families of locally supported strictly positive definite zonal functions, essentially by the self-convolution of the characteristic functions of spherical caps. This is the analog for the sphere of the construction of the circular and spherical covariances for $\mathbb{R}^2$ and $\mathbb{R}^3$, and more generally of the construction of Euclid's hat functions (see Wu [20] and Gneiting [9]). See Ziegel [22] and the references there for related work in the statistics community on generating positive definite functions on the sphere via self-convolution.

In what follows let $\theta(x, y) = \arccos(x^T y)$ denote the geodesic distance on $\mathbb{S}^d$.

Definition 1.1. A continuous function $g : [0, \pi] \to \mathbb{R}$ is (zonal) positive definite on the sphere $\mathbb{S}^d$ if for all distinct point sets $X = \{x_1, \ldots, x_n\}$ on the sphere and all $n \in \mathbb{N}$, the matrices $M_X := \left[g(\theta(x_i, x_j))\right]_{i,j=1}^n$ are positive semi-definite, that is, $\boldsymbol{c}^T M_X \boldsymbol{c} \geq 0$ for all $\boldsymbol{c} \in \mathbb{R}^n$. The function g is *(zonal) strictly positive definite* on $\mathbb{S}^d$ if the matrices are all positive definite, that is, $\boldsymbol{c}^T M_X \boldsymbol{c} > 0$, for all nonzero $\boldsymbol{c} \in \mathbb{R}^n$. The notation Ψ_d will denote the cone of all positive definite functions on $\mathbb{S}^d$ and Ψ_d^+ the subcone of all strictly positive definite functions on $\mathbb{S}^d$. Λ_d will denote the cone of all functions $f \in C[-1, 1]$ such that $f \circ \cos \in \Psi_d$. Λ_d^+ will denote the cone of all functions $f \in C[-1, 1]$ such that $f \circ \cos \in \Psi_d^+$.

In what follows we abuse notation somewhat by referring to $f : [-1, 1] \to \mathbb{R}$ as a zonal function when it is $f \circ \cos : [0, \pi] \to \mathbb{R}$ which is the zonal function. In the same spirit we will refer to Λ_d and Λ_d^+ as cones of positive definite and strictly positive definite functions, even though strictly speaking the relevant cones consist of the zonal functions $\Lambda_d \circ \cos$ and $\Lambda_d^+ \circ \cos$.

Zonal positive definite functions (radial basis functions on the sphere) have been used for interpolation or approximation of scattered data on the sphere (see [7,8] and the references therein). The standard model in this setting is a linear combination of translates (rotations) of the *zonal basis function*. Thus, the interpolation problem is

Problem 1.2. Given a zonal function g, n distinct points $x_i \in \mathbb{S}^d$ and n corresponding values $f_i \in \mathbb{R}$, find coefficients $c_j \in \mathbb{R}$ such that

$$s(x) = \sum_{j=1}^{n} c_j g(\theta(x, x_j)), \quad x \in \mathbb{S}^d,$$

satisfies

$$s(x_i) = f_i, \quad 1 \leq i \leq n.$$

Strict positive definiteness of g is exactly the condition needed to guarantee that this interpolation problem has a unique solution irrespective of the position of the nodes $\{x_i\}$. Thus, for interpolation by a weighted sum of rotations of g, strict positive definiteness of g is critical.

Zonal positive definite functions are also of considerable importance in statistics where they serve as covariance models. In this statistical context positive definiteness is essential but strict positive definiteness is not. Gneiting [10] gives an excellent survey of positive definite functions on spheres from a statistical point of view.

It was Schoenberg [16] who characterized the class of positive definite, zonal functions on the sphere proving the following. His result is stated in terms of the Gegenbauer polynomials $\{C_n^\lambda\}$, the family for fixed $\lambda \geq 0$ being orthogonal with respect to the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$. Schoenberg shows

Theorem 1.3. *Consider a continuous function f on $[-1,1]$. The function $f \circ \cos$ is a positive definite function on $\mathbb{S}^d$, i.e. $f \in \Lambda_d$, if and only if f has a Gegenbauer expansion*

$$f(x) \sim \sum_{n=0}^{\infty} a_n C_n^\lambda(x), \tag{1.1}$$

$\lambda = (d-1)/2$*, in which all the coefficients a_n are nonnegative and in which the series converges at $x = 1$.*

Since $\max_{x\in[-1,1]} |C_n^\lambda(x)| = C_n^\lambda(1)$ (for the normalization see (4.2)) the Weierstrass M-test implies that the series with nonnegative coefficients (1.1) converges at $x = 1$ if and only if it converges uniformly on $[-1,1]$.

The characterization of strictly positive definite functions on $\mathbb{S}^d$ came somewhat later. A simple sufficient condition of Xu and Cheney [21] states that $f \circ \cos$ is strictly positive definite on $\mathbb{S}^d$, i.e. $f \in \Lambda_d^+$, if in addition to the conditions of Theorem 1.3, all the Gegenbauer coefficients a_n of f in expansion (1.1) are positive. Chen, Menegatto and Sun [4] showed that a necessary and sufficient condition for $f \circ \cos$ to be strictly positive definite on $\mathbb{S}^d$, $d \geq 2$, is that, in addition to the conditions of Theorem 1.3, infinitely many of the Gegenbauer coefficients with odd index, and infinitely many of those with even index, are positive. The Chen, Menegatto and Sun criteria is necessary but not sufficient for strict positive definiteness on $\mathbb{S}^1$ (see [4, p. 2740]). Menegatto, Oliveira and Peron [14] give necessary and sufficient conditions for a zonal kernel to be strictly positive definite on $\mathbb{S}^1$.

The following notations will be used throughout the paper. C_n^λ is the degree n member of the family of Gegenbauer polynomials with parameter λ, $\{C_n^\lambda\}_{n=0}^\infty$, whose members are orthogonal with respect to the weight$(1-x^2)^{\lambda-\frac{1}{2}}$ on $[-1,1]$. Abramowitz and Stegun [1] is an excellent reference for the properties of these functions. The normalization of the C_n^λ's adopted here is that used in [1]. The Chebyshev polynomials of the first and second kind will be written as $\{T_n\} = \{\frac{n}{2}C_n^0\}$ and $\{U_n\} = \{C_n^1\}$, respectively. Define the cone of CMS functions, $\Lambda_d^{\boxplus}$, to be the set of functions $f \in \Lambda_d$ such that infinitely many of the Gegenbauer coefficients of odd index and infinitely many of the Gegenbauer coefficients of even index are positive. Define the cone of CX functions, $\Lambda_d^{\blacksquare}$, to be the cone of nonnegative functions in Λ_d such that all the Gegenbauer coefficients are positive. The results of Chen, Menegatto and Sun, and of Xu and Cheney, imply the relationships between the cones just defined,

$$\Lambda_m^{\blacksquare} \subset \Lambda_m^+ = \Lambda_m^{\boxplus} \subset \Lambda_m \quad \text{when } m \geq 2 \quad \text{and} \quad \Lambda_1^{\blacksquare} \subset \Lambda_1^+ \subset \Lambda_1^{\boxplus} \subset \Lambda_1, \tag{1.2}$$

which will be very important throughout the rest of the paper.

The paper is laid out as follows. The montée and descente operators for the sphere will be discussed in Section 2. The main results in this section concern the positive definiteness preserving properties of these dimension hopping operators. In Section 3 the montée operator is used to derive families of strictly positive definite functions of increasing smoothness from known strictly positive definite functions. This is the analog for the sphere of the construction of the Wendland functions for $\mathbb{R}^d$ from the Askey functions $(1-r)_+^\ell$. Section 4 will consider convolution structures for the Gegenbauer polynomials. The main result in the section, Theorem 4.1, shows how convolution of two zonal functions for $\mathbb{S}^{d+2}$ can be performed indirectly by performing a simpler convolution two dimensions below. In Section 5 a second family of strictly positive definite zonal functions is developed. The functions essentially arise from the convolution of the characteristic functions of spherical caps, and therefore are analogs for the sphere of the circular and spherical covariances of the Euclidean case.

2. Montée and descente on spheres

This section considers some dimension hopping operators for spheres $\mathbb{S}^d$. These have properties akin to those of the dimension hopping montée and descente operators of Matheron [13, section 1.3.3] for radial functions on $\mathbb{R}^d$. Broadly speaking the montée operator $\mathcal{I}$ increases smoothness and maps (strictly) positive definite functions for $\mathbb{S}^{d+2}$ to (strictly) positive definite functions for $\mathbb{S}^d$. Its inverse, the descente operator $\mathcal{D}$, decreases smoothness and maps (strictly) positive definite functions for $\mathbb{S}^d$ to (strictly) positive definite functions for $\mathbb{S}^{d+2}$. More precise statements will be given below.

Definition 2.1. Given f absolutely continuous on $[-1,1]$ define $\mathcal{D}f$ by

$$(\mathcal{D}f)(x) = f'(x), \quad x \in [-1,1]. \tag{2.1}$$

Also, given f integrable on $[-1,1]$ define an operator $\mathcal{I}$ by

$$(\mathcal{I}f)(x) = \int_{-1}^{x} f(u)\,du. \tag{2.2}$$

Recall from elementary analysis that if $f \in L^1[-1,1]$ then $\mathcal{I}f$ is absolutely continuous on $[-1,1]$ and

$$(\mathcal{D}\mathcal{I}f)(x) = f(x), \quad \text{for almost every } x \in [-1,1].$$

In the other direction, if f is absolutely continuous on $[-1,1]$, then f is almost everywhere differentiable on $[-1,1]$ and the derivative is integrable with

$$(\mathcal{I}\mathcal{D}f)(x) = f(x) - f(-1), \quad \text{for all } x \in [-1,1].$$

If we are considering $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ then the relevant Gegenbauer index is $\lambda = (d-1)/2$.

The reader will recall that formulas involving Gegenbauer polynomials with index $\lambda = 0$ have to be understood in a limiting sense as

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda} C_n^\lambda(x) = C_n^0(x) = \begin{cases} \frac{2}{n} T_n(x), & n > 0, \\ 1, & n = 0. \end{cases}$$

We now turn to questions of the preservation of positive definiteness under the action of the operators $\mathcal{I}$ and $\mathcal{D}$. Then [17, 4.7.14]

$$\mathcal{D}C_n^\lambda = \begin{cases} 2\lambda\, C_{n-1}^{\lambda+1}, & \lambda > 0, \\ 2\, C_{n-1}^{1} = 2U_{n-1}, & \lambda = 0. \end{cases}$$

It will be useful to define an auxiliary index μ by

$$\mu_\lambda = \begin{cases} \lambda, & \lambda > 0, \\ 1, & \lambda = 0. \end{cases} \tag{2.3}$$

Using the μ_λ notation the relationship above takes the compact form

$$\mathcal{D}C_n^\lambda = 2\mu_\lambda\, C_{n-1}^{\lambda+1}, \quad \lambda \geq 0. \tag{2.4}$$

In terms of $\mathcal{I}$ Eq. (2.4) becomes

$$\mathcal{I}C_{n-1}^{\lambda+1} = \frac{1}{2\mu_\lambda}\left(C_n^\lambda - C_n^\lambda(-1)\right), \quad \lambda \geq 0. \tag{2.5}$$

The following theorems show that the *montée* operator $\mathcal{I}$ maps positive definite functions $f \circ \cos$ on $\mathbb{S}^{d+2}$ into smoother positive definite functions $(\mathcal{I} f) \circ \cos$ on $\mathbb{S}^d$. Furthermore, the *descente* operator $\mathcal{D}$ maps positive definite functions $f \circ \cos$ on $\mathbb{S}^d$ into rougher positive definite functions $(\mathcal{D} f) \circ \cos$ on $\mathbb{S}^{d+2}$, unless $\mathcal{D} f$ either fails to exist, or fails to be continuous. Eqs. (2.4) and (2.5) already show these positive definiteness preserving properties for the positive definite spherical functions $C_n^\nu \circ \cos$.

The same results almost hold for strictly positive definite functions, only the results involving $\mathcal{S}^1$ being slightly different. The statements concerning strict positive definiteness are most clearly set out in terms of the cone of CMS functions $\Lambda_m^{\boxplus}$, and the cone of CX functions $\Lambda_m^{\blacksquare}$, (see (1.2) above).

Theorem 2.2. *Let $m \in \mathbb{N}$.*

(a) (i) *If $f \in \Lambda_{m+2}$ then there is a constant C such that $C + \mathcal{I} f \in \Lambda_m$.*
(ii) *If $f \in \Lambda_{m+2}^+$ then there is a constant C such that $C + \mathcal{I} f \in \Lambda_m^{\boxplus}$.*
(iii) *If, in addition, f is nonnegative then the constant C in parts* (i) *and* (ii) *can be chosen as zero.*

(b) *Let $f \in \Lambda_{m+2}^{\blacksquare}$ then $\mathcal{I} f \in \Lambda_m^{\blacksquare}$.*

Theorem 2.3. *Suppose that $f \in \Lambda_m$, $m \geq 1$, has derivative $f' \in C[-1, 1]$. Then $\mathcal{D} f \in \Lambda_{m+2}$. If, in addition, $f \in \Lambda_m^{\boxplus}$ then $\mathcal{D} f \in \Lambda_{m+2}^+$.*

Note that in this theorem the explicit assumptions on f are weak, principally that f' is continuous. There is no need to assume for f' the greater amount of smoothness necessary to guarantee a general function has a uniformly convergent Gegenbauer series $\sum_{n=0}^{\infty} c_n C_n^{\lambda+1}$.

2.1. Proofs of the results concerning positive definiteness and the dimension hopping operators $\mathcal{I}$ and $\mathcal{D}$

Proof of Theorem 2.2. *Proof of* (a)(i): Since $f \in \Lambda_{m+2}$, it follows from Theorem 1.3 that

$$f(x) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+1}(x),$$

where all the coefficients a_n are nonnegative, and the series is absolutely and uniformly convergent for all $x \in [-1, 1]$. Integrating term by term using the boundedness of the operator $\mathcal{I}$ and (2.5) we obtain another uniformly convergent series

$$(\mathcal{I} f)(x) = \sum_{n=0}^{\infty} b_n C_n^\lambda(x), \quad x \in [-1, 1]. \tag{2.6}$$

According to (2.5) the coefficient b_n has the same sign as the coefficient a_{n-1}. Hence, for a suitable constant C, $C + \mathcal{I} f$ has nonnegative Gegenbauer coefficients. Applying Theorem 1.3 again it follows that $C + \mathcal{I} f$ is in Λ_m.

Proof of (a)(ii): From the definition of the cone of CMS functions and since $\Lambda_{m+2}^{+} = \Lambda_{m+2}^{\boxplus}$ by (1.2), part (a)(ii) follows by almost exactly the same argument as part (a)(i).

Proof of (a)(iii): The nonnegativity of f in $[-1, 1]$ implies $\mathcal{I} f$ is also nonnegative. Since f is nontrivial it follows that $\mathcal{I} f$ is nontrivial. Since the constant part of the Gegenbauer series expansion of $\mathcal{I} f$ is a weighted average, with weight function $\left(1 - x^2\right)^{\lambda - \frac{1}{2}}$, of the values of $(\mathcal{I} f)(x)$, $-1 < x < 1$, it follows that this constant is positive. The conclusion follows.

Proof of (b): Recall that the cone $\Lambda_m^{\blacksquare}$ is the set of all functions f on $[-1, 1]$ that are nonnegative, belong to Λ_m, and have all the Gegenbauer coefficients positive. Assume now $f \in \Lambda_{m+2}^{\blacksquare}$. The Xu and Cheney criterion then implies $f \in \Lambda_{m+2}^{+}$. Since f is strictly positive definite it must be nontrivial. The argument of part (a)(i) shows that the series (2.6) converges uniformly on $[-1, 1]$, and that all the coefficients b_n with $n > 0$ are positive. The positivity of the constant part in the expansion of $\mathcal{I} f = \sum_{n=0}^{\infty} b_n C_n^\lambda$ then follows as in the proof of (a)(iii). The nonnegativity of $\mathcal{I} f$ on $[-1, 1]$ follows from that of f. Therefore, $\mathcal{I} f \in \Lambda_m^{\blacksquare} \subset \Lambda_m^{+}$, as required. □

The following lemma shows that the coefficients of the (formal) Gegenbauer series for the derivative f' can be calculated term by term from the coefficients in the (formal) Gegenbauer series for f.

Lemma 2.4. *Let f be an absolutely continuous function on $[-1, 1]$. Suppose f and f' have (formal) Gegenbauer series*

$$f \sim \sum_{n=0}^{\infty} a_n C_n^\lambda \quad \text{and} \quad f' \sim \sum_{n=0}^{\infty} b_n C_n^{\lambda+1}, \text{ respectively.}$$

Then, for $n \in \mathbb{N}$,

$$b_{n-1} = 2\mu_\lambda a_n, \quad \lambda \geq 0.$$

Proof. Let $\widetilde{b_{n-1}} = h_{n-1}^{\lambda+1} b_{n-1}$ where $h_{n-1}^{\lambda+1} = \int_{-1}^{1} \left(C_{n-1}^{\lambda+1}(x)\right)^2 (1 - x^2)^{\lambda+\frac{1}{2}}\, dx$. Then proceed by integration by parts:

$$\begin{aligned}
\widetilde{b_{n-1}} &= \int_{-1}^{1} C_{n-1}^{\lambda+1}(x)(1 - x^2)^{\lambda+\frac{1}{2}}\, f'(x)\, dx \\
&= \int_{-1}^{1} f(x)(1 - x^2)^{\lambda-\frac{1}{2}} \left\{(2\lambda + 1)x C_{n-1}^{\lambda+1}(x) - (1 - x^2)\frac{d}{dx} C_{n-1}^{\lambda+1}(x)\right\} dx.
\end{aligned} \tag{2.7}$$

Applying formula [1, (22.8.2)] the expression in curly braces above becomes

$$(2\lambda + 1)x C_{n-1}^{\lambda+1}(x) - (1 - x^2)\frac{d}{dx} C_{n-1}^{\lambda+1}(x) = (2\lambda + n)\left(x C_{n-1}^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x)\right). \tag{2.8}$$

Then a combination of the three term recurrence relation (see [1, (22.7.3)]), and the recurrence on order formula [1, (22.7.23)], shows that

$$xC_{n-1}^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x) = \frac{n}{2\lambda} C_n^{\lambda}, \quad \lambda > 0. \tag{2.9}$$

Using (2.8) and (2.9) to rewrite (2.7) yields

$$\widetilde{b_{n-1}} = \frac{n(2\lambda + n)}{2\lambda} \int_{-1}^{1} f(x) C_n^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}}\, dx = \frac{n(2\lambda+n)}{2\lambda}\widetilde{a_n}, \quad \lambda > 0, \tag{2.10}$$

where $\widetilde{a_n} = h_n^{\lambda} a_n$. From [1, (22.2.3)]

$$h_n^{\lambda} = \frac{\pi \Gamma(n+2\lambda)}{2^{2\lambda-1} n!(n+\lambda)\Gamma^2(\lambda)}, \quad \lambda > 0.$$

Substituting into (2.10)

$$b_{n-1} = \frac{\widetilde{b_{n-1}}}{h_{n-1}^{\lambda+1}} = \frac{h_n^{\lambda}}{h_{n-1}^{\lambda+1}} \frac{n(2\lambda+n)}{2\lambda} a_n = 2\lambda a_n, \quad \lambda > 0, \text{ and } n \in \mathbb{N}. \tag{2.11}$$

This is the result for $\lambda > 0$. Recall that $C_n^0(x) = \lim_{\lambda\to 0^+} \frac{C_n^{\lambda}(x)}{\lambda}$. Hence, in the obvious notation, the Gegenbauer coefficient $a_n^0 = \lim_{\lambda \to 0^+} \lambda a_n^{\lambda}$. Thus, the result for $\lambda = 0$ follows from Eq. (2.11). □

Proof of Theorem 2.3. Let $\lambda = (m-1)/2$ and f and f' have Gegenbauer series

$$f \sim \sum_{n=0}^{\infty} a_n C_n^{\lambda} \quad \text{and} \quad f' \sim \sum_{n=0}^{\infty} b_n C_n^{\lambda+1}.$$

Since $f \in \Lambda_m$ Schoenberg's characterization implies that all the coefficients a_n are nonnegative and the series for f converges uniformly on $[-1, 1]$.

It follows from Lemma 2.4 that all the b_n's are also nonnegative. Szegö [17, Theorem 9.1.3] gives a result concerning Cesàro summability that implies that the Gegenbauer series of any function $g \in C[-1, 1]$ is Abel summable at $x = 1$ to $g(1)$. Applying this result to f' we see that the series $\sum_{n=0}^{\infty} b_n C_n^{\lambda+1}(1)$ is Abel summable to $f'(1)$. But this is a series of nonnegative terms, hence the Abel summability implies summability. Since $\left|C_n^{\lambda+1}(x)\right| \le C_n^{\lambda+1}(1)$, for all $x \in [-1, 1]$, it follows that the Gegenbauer series of f' converges uniformly by the Weierstrass M-test. The well know theorem about the uniform convergence of a term by term derivative series then shows that this series converges uniformly to f'. An application of the Schoenberg characterization, Theorem 1.3, now shows that $f' \circ \cos \in \Psi_{m+2}$, completing the proof of the first part of the proposition.

Turn now to the second claim in the theorem. Assume $f \in \Lambda_m^{\boxplus}$. From the first part of the theorem $f' \in \Lambda_{m+2}$. Then, from the definition of the cone of CMS functions and since by Lemma 2.4 b_{n-1} has the same sign as a_n, it follows that $f' \in \Lambda_{m+2}^{\boxplus} = \Lambda_{m+2}^{+}$. □

3. Positive definite functions generated from the basic functions of the Pólya criteria for $\mathbb{S}^d$

In this section the montée operator $\mathcal{I}$ is used to construct families of strictly positive definite zonal functions of increasing smoothness starting from less smooth parent functions known to be strictly positive definite.

The construction of this section starts from the locally supported zonal functions $(t-\theta)_+^\mu$, $0<t<\pi$, known to be strictly positive definite on $\mathbb{S}^{2\mu-1}$ with all Gegenbauer coefficients positive, for $2\le\mu\le 4$ (see [3]). These functions were conjectured to be strictly positive definite on the corresponding sphere for all integers $\mu\ge 2$.[1] The constructions discussed in this section are the analog for the sphere of the construction of the Wendland functions [19] for $\mathbb{R}^d$.

The proof in [3] shows that $f_m\in\Lambda_{2m-1}^{\blacksquare}\subset\Lambda_{2m-1}^{+}$, for $2\le m\le 4$. The construction starts with the case $\mu=2$ of the function f_μ,

$$f_2(\cos\theta)=g_2(\theta):=(t-\theta)_+^2,\quad 0<t<\pi.$$

Calculating

$$(\mathcal{I}f_2)(\cos\theta)=\begin{cases}\cos(\theta)\left((t-\theta)^2-2\right)+2\sin(\theta)(t-\theta)+2\cos(t), & 0\le\theta<t,\\ 0, & t\le\theta\le\pi.\end{cases}$$

From Theorem 2.2(b) since $f_2\in\Lambda_3^{\blacksquare}$ it follows that $\mathcal{I}f_2\in\Lambda_1^{\blacksquare}\subset\Lambda_1^{+}$.

Next consider the case $\mu=3$. Then

$$f_3(\cos\theta)=g_3(\theta):=(t-\theta)_+^3,\quad 0<t<\pi.$$

Applying the $\mathcal{I}$ operator, and writing $u=t-\theta$,

$$(\mathcal{I}f_3)(\cos\theta)=\begin{cases}\cos(\theta)(u^3-6u)+\sin(\theta)(3u^2-6)+6\sin(t), & 0\le\theta<t,\\ 0, & t\le\theta\le\pi.\end{cases}$$

and

$$\begin{aligned}&(\mathcal{I}^2f_3)(\cos\theta)\\ &\quad=\cos(2\theta)(a_7u^3+a_6u)+\sin(2\theta)(a_5u^2+a_4)+\cos(\theta)a_3+a_2u^3+a_1u+a_0,\end{aligned}$$

for $0\le\theta<t$, and equals 0 when $t\le\theta\le\pi$. Here,

$$a_7=\frac{1}{4},\quad a_6=-\frac{21}{8},\quad a_5=\frac{9}{8},\quad a_4=-\frac{45}{16},\quad a_3=6\sin(t),$$
$$a_2=\frac{1}{2},\quad a_1=-3,\quad\text{and}\quad a_0=-\frac{3}{16}\sin(2t).$$

Applying Theorem 2.2(b) again, since f_3 is in $\Lambda_5^{\blacksquare}$ it follows that $\mathcal{I}f_3\in\Lambda_3^{\blacksquare}\subset\Lambda_3^{+}$ and that $\mathcal{I}^2f_3\in\Lambda_1^{\blacksquare}\subset\Lambda_1^{+}$. Note that in evaluating the function $\mathcal{I}^2f_3$, and other functions yet to be constructed, efficiency gains can clearly be made by rearranging expressions, precomputing coefficients, and using nested multiplication.

Also

$$f_4(\cos\theta)=g_4(\theta):=(t-\theta)_+^4,\quad 0<t<\pi.$$

Applying the $\mathcal{I}$ operator, and writing $u=t-\theta$,

$$\begin{aligned}&(\mathcal{I}f_4)(\cos\theta)\\ &\quad=\begin{cases}\cos(\theta)(u^4-12u^2+24)+\sin(\theta)(4u^3-24u)-24\cos(t), & 0\le\theta<t,\\ 0, & t\le\theta\le\pi.\end{cases}\end{aligned}$$

[1] Note added in proof: Yuan Xu has recently given a proof of this conjecture in his paper “Positive definite functions on the unit sphere and integrals of Jacobi polynomials”, arXiv: 1701.00787 [math.CA] (2017).

and

$$(\mathcal{I}^2 f_4)(\cos\theta) = \cos(2\theta)\left(b_8u^4 + b_7u^2 + b_6\right) + \sin(2\theta)\left(b_5u^3 + b_4u\right) + \cos(\theta)b_3 + \left(b_2u^4 + b_1u^2 + b_0\right),$$

for $0 \le \theta < t$, and equals 0 when $t \le \theta \le \pi$. Here,

$$b_8 = \frac{1}{4}, \quad b_7 = -\frac{21}{4}, \quad b_6 = \frac{93}{8}, \quad b_5 = \frac{3}{2}, \quad b_4 = -\frac{45}{4},$$
$$b_3 = -24\cos(t), \quad b_2 = \frac{1}{2}, \quad b_1 = -6,$$

and $b_0 = \frac{3}{4}\cos^2(t) + \frac{93}{8}$. From Theorem 2.2(b) again, since f_4 is a $C[-1,1]$ function in $\Lambda_7^{\blacksquare}$ it follows that $\mathcal{I}f$ is a $C^1[-1,1]$ function in $\Lambda_5^{\blacksquare} \subset \Lambda_5^+$, that $\mathcal{I}^2 f_4$ is a $C^2[-1,1]$ function in $\Lambda_3^{\blacksquare} \subset \Lambda_3^+$, and that $\mathcal{I}^3 f_4$ is a $C^3[-1,1]$ function in $\Lambda_1^{\blacksquare} \subset \Lambda_1^+$.

For the practically important special case of approximation on $\mathbb{S}^2$ or $\mathbb{S}^3$ the construction above yields a list of locally supported functions in Λ_3^+ of increasing smoothness, namely $f_2 \in C[-1,1]$, $\mathcal{I} f_3 \in C^1[-1,1]$ and $\mathcal{I}^2 f_4 \in C^2[-1,1]$.

For approximation on $\mathbb{S}^1$ the construction yields the following list of locally supported functions in Λ_1^+, $f_2 \in C[-1,1]$, $\mathcal{I} f_2 \in C^1[-1,1]$, $\mathcal{I}^2 f_3 \in C^2[-1,1]$, and $\mathcal{I}^3 f_4 \in C^3[-1,1]$.

Should the conjecture of [3] be proven, as it now has[1] then the construction of positive definite families by the method of this section could easily be extended. For example a double integration by parts establishes the recurrence formula

$$(\mathcal{I} f_m)(\cos(\theta)) = \cos(\theta)(t-\theta)_+^m + m\sin(\theta)(t-\theta)_+^{m-1} - m(m-1)(\mathcal{I} f_{m-2})(\cos(\theta)),$$

where $f_m(\cos(\theta)) = (t-\theta)_+^m$, $0 < t < \pi$, $m \in \mathbb{N}$, which together with the initial values

$$(\mathcal{I} f_1)(\cos(\theta)) = \begin{cases} \cos(\theta)(t-\theta) + \sin(\theta) - \sin(t), & 0 \le \theta < t, \\ 0, & t \le \theta \le \pi, \end{cases}$$

and

$$(\mathcal{I} f_2)(\cos(\theta)) = \cos(\theta)(t-\theta)_+^2 + 2\sin(\theta)(t-\theta)_+ - 2(\cos(\theta) - \cos(t))_+,$$

enables computation of $(\mathcal{I} f_m)(\cos\theta)$ for all positive integers m.

Finally, note that the functions $f_\mu(\cos\theta) = (t-\theta)_+^\mu$, $2 \le \mu < \infty$, provide an alternative family of locally supported, strictly positive definite functions of increasing smoothness on $\mathbb{S}^3$.

4. Convolution via dimension hopping

This section concerns a connection between the dimension hopping operators $\mathcal{D}$ and $\mathcal{I}$, and certain convolution structures for Gegenbauer expansions. The main result, Theorem 4.1, shows that the convolution of two zonal functions for $\mathbb{S}^{d+2}$ can be calculated indirectly via the convolution of related zonal functions for the sphere $\mathbb{S}^d$.

The notation $\star_\lambda$ will be used to denote a convolution associated with Gegenbauer series in the polynomials $\{C_n^\lambda\}_{n=0}^\infty$. As explained in Section 4.1 it is naturally associated with a convolution of zonal functions on $\mathbb{S}^{2\lambda+1}$.

Theorem 4.1. *Let f and g be functions in $L^1[-1,1]$ and $(\mathcal{I}f)\star_\lambda(\mathcal{I}g)$ be absolutely continuous. Then*

$$(f\star_{\lambda+1} g)(x) = (2\lambda+1)\mathcal{D}\Big[(\mathcal{I}f)\star_\lambda(\mathcal{I}g)\Big](x), \quad \textit{almost everywhere in } [-1,1]. \tag{4.1}$$

Theorem 4.1 will be applied to construct a family of strictly positive definite zonal functions in Section 5.

4.1. A convolution structure for the Gegenbauer polynomials

In this section it is convenient to use a different normalization of the Gegenbauer polynomials, one in which the Gegenbauer expansion and the associated convolution take particularly simple form. Namely, normalize so that the orthogonal polynomials are one at one, taking $W_n^\lambda(x) = C_n^\lambda(x)/C_n^\lambda(1)$, where

$$C_n^\lambda(1) = \begin{cases} \begin{pmatrix} n+2\lambda-1 \\ n \end{pmatrix} = \dfrac{\Gamma(2\lambda+n)}{\Gamma(2\lambda)\Gamma(n+1)}, & \lambda>0,\ n>0, \\ \dfrac{2}{n}, & \lambda=0,\ n>0. \end{cases} \tag{4.2}$$

Set $\Omega_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}$. The orthogonality in terms of this W_n^λ normalization is

$$\int_{-1}^{1} W_n^\lambda(x) W_m^\lambda(x)\,\Omega_\lambda(x)dx = \frac{1}{w_\lambda(n)}\,\delta_{nm}, \quad n,m\in\mathbb{N}_0,$$

where

$$w_\lambda(n) = \begin{cases} \dfrac{\Gamma(\lambda)\,(n+\lambda)\,\Gamma(n+2\lambda)}{\pi^{1/2}\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(2\lambda)\Gamma(n+1)}, & \lambda>0,\ n\in\mathbb{N}_0, \\ 2/\pi, & \lambda=0,\ n\in\mathbb{N}, \\ 1/\pi, & \lambda=0,\ n=0. \end{cases}$$

Now for $f\in L_1([-1,1],\Omega_\lambda)$ define Fourier–Gegenbauer coefficients as

$$\widehat{f_\lambda}(n) = \int_{-1}^{1} f(x) W_n^\lambda(x)\,\Omega_\lambda(x)dx, \quad n\in\mathbb{N}_0.$$

Then the formal series expansion can be written in terms of the W_n^λ's as

$$f \sim \sum_{n=0}^{\infty} w_\lambda(n)\widehat{f_\lambda}(n) W_n^\lambda. \tag{4.3}$$

From the definition of the Fourier coefficient and the orthogonality it follows immediately that

$$\widehat{\left(W_m^\lambda\right)}_\lambda(n) = \frac{\delta_{nm}}{w_\lambda(n)}. \tag{4.4}$$

Associated with the Gegenbauer series is a convolution $\star_\lambda$. This convolution is based upon the product relation due to Gegenbauer

$$\int_{-1}^{1} W_n^\lambda(x) C_\lambda(x,y,z)\,\Omega_\lambda(x)dx = W_n^\lambda(y)\,W_n^\lambda(z), \quad \lambda>0.$$

Note that Hirschman [11] provides an explicit form of the density $C_\lambda(x, y, z)$ showing that the function vanishes if $1 - x^2 - y^2 - z^2 + 2xyz \le 0$.

The convolution $\star_\lambda$ is defined in terms of a generalized translation [11] as

$$(f \star_\lambda g)(x) = \int_{-1}^{1}\int_{-1}^{1} f(y)g(z)C_\lambda(x, y, z)\,\Omega_\lambda(y)\,\Omega_\lambda(z)dy\,dz,$$

when $\lambda > 0$. When $\lambda = 0$ it may be defined by

$$(f \star_0 g)(\cos\theta) = \frac{1}{2}\int_{-\pi}^{\pi} f(\cos(\theta - t))\, g(\cos(t))\, dt. \tag{4.5}$$

The latter definition may be viewed as going over to the circle with the substitution $x = \cos\theta$, convolving there and coming back, as is commonly done in proofs of Jackson theorems for algebraic polynomial approximation.

The convolution has the properties listed in the theorem below. Hirschman [11] gives proofs of these properties when $\lambda > 0$. The parts concerning the special case $\lambda = 0$ have been added as they are needed later.

Theorem 4.2. *Let $f, g, h \in L_1([-1, 1], \Omega_\lambda)$. Then*

(i) $\|f \star_\lambda g\| \le \|f\|\,\|g\|$.
(ii) $f \star_\lambda g = g \star_\lambda f$.
(iii) $f \star_\lambda (g \star_\lambda h) = (f \star_\lambda g) \star_\lambda h$.
(iv) $\widehat{[f \star_\lambda g]}_\lambda(n) = \widehat{f}_\lambda(n)\,\widehat{g}_\lambda(n)$, *for all* $n \in \mathbb{N}_0$.

It is well known that the convolution for functions in $L_1([-1, 1], \Omega_\lambda)$ described above is equivalent to convolution of zonal functions on the sphere. For example Dunkl [5] writes

> "the space of zonal functions on $\mathbb{S}^k$ is isomorphic to $L_1([1, 1], \Omega_{(k-1)/2})$ and the "spherical convolution" is isomorphic to Bochner's".

Here, the convolution of Bochner being referenced by Dunkl is the one that Hirschman and we are using. Explicitly, defining

$$(f \circledast_d g)(\langle u, v\rangle) := \int_{\mathbb{S}^d} f(\langle u, w\rangle)g(\langle w, v\rangle)\,d\sigma(w) \tag{4.6}$$

one can show

$$f \star_\lambda g = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{2\pi^{\lambda+\frac{1}{2}}}\, f \circledast_{2\lambda+1} g, \quad \lambda \in \mathbb{N}_0. \tag{4.7}$$

4.2. Proof of Theorem 4.1

Write F for $\mathcal{I} f$ and G for $\mathcal{I} g$. Then f has a Gegenbauer series

$$f \sim \sum_{n=0}^{\infty} w_{\lambda+1}(n)\widehat{f}_{\lambda+1}(n)W_n^{\lambda+1}(x) = \sum_n \left(\frac{w_{\lambda+1}(n)\widehat{f}_{\lambda+1}(n)}{C_n^{\lambda+1}(1)}\right) C_n^{\lambda+1}(x),$$

and F has Gegenbauer series

$$F \sim \sum_{n=0}^{\infty} w_\lambda(n)\widehat{F}_\lambda(n)W_n^\lambda(x) = \sum_{n=0}^{\infty}\left(\frac{w_\lambda(n)\widehat{F}_\lambda(n)}{C_n^\lambda(1)}\right)C_n^\lambda(x).$$

Since $f \in L^1[-1,1]$, F is absolutely continuous. Therefore applying Lemma 2.4

$$\frac{w_{\lambda+1}(n)\widehat{f}_{\lambda+1}(n)}{C_n^{\lambda+1}(1)} = 2\mu_\lambda \frac{w_\lambda(n+1)\widehat{F}_\lambda(n+1)}{C_{n+1}^\lambda(1)}, \quad n \geq 0.$$

Hence,

$$\widehat{F}_\lambda(n+1) = a_{\lambda,n+1}\,\widehat{f}_{\lambda+1}(n), \quad n \geq 0,$$

where

$$a_{\lambda,n+1} = \frac{1}{2\mu_\lambda}\frac{C_{n+1}^\lambda(1)}{C_n^{\lambda+1}(1)}\frac{w_{\lambda+1}(n)}{w_\lambda(n+1)}.$$

Similarly, $\widehat{G}_\lambda(n+1) = a_{\lambda,n+1}\,\widehat{g}_{\lambda+1}(n)$, for $n \geq 0$. Since F and G are absolutely continuous $F \star_\lambda G$ is well defined with

$$(F \star_\lambda G)(x) \sim \sum_{n=0}^{\infty} w_\lambda(n)\widehat{F}_\lambda(n)\widehat{G}_\lambda(n)W_n^\lambda(x) \sim \sum_{n=0}^{\infty}\frac{w_\lambda(n)\widehat{F}_\lambda(n)\widehat{G}_\lambda(n)}{C_n^\lambda(1)}C_n^\lambda(x).$$

Since $F \star_\lambda G$ is absolutely continuous another application of Lemma 2.4 shows

$$\begin{aligned}\mathcal{D}(F \star_\lambda G)(x) &\sim 2\mu_\lambda \sum \frac{w_\lambda(n+1)\widehat{F}_\lambda(n+1)\widehat{G}_\lambda(n+1)}{C_{n+1}^\lambda(1)}C_n^{\lambda+1}(x)\\ &\sim \sum_n a_{\lambda,n+1}w_{\lambda+1}(n)\widehat{f}_{\lambda+1}(n)\widehat{g}_{\lambda+1}(n)W_n^{\lambda+1}(x).\end{aligned}$$

Now a calculation shows that $a_{\lambda,n+1} = \frac{1}{2\lambda+1}$ for all $\lambda \geq 0$, and all nonnegative integers n.

Hence from the convolution rule in Theorem 4.2 part (iv), functions $f \star_{\lambda+1} g$ and $(2\lambda+1)\mathcal{D}((\mathcal{I}f)\star_\lambda(\mathcal{I}g))$ have the same Gegenbauer coefficients.

However, a consequence of Kogbetliantz's result [12] concerning the positivity (in the operator sense) of the Cesàro means of order $2\lambda+1$, $\{\sigma_N^{2\lambda+1}h\}$, of a function $h \in L_1([-1,1],\Omega_\lambda)$, is that $\sigma_N^{2\lambda+1}(h) \to h$, in the sense of $L_1([-1,1],\Omega_\lambda)$, as $N \to \infty$. This in turn implies the well known uniqueness theorem that a function $h \in L_1([-1,1],\Omega_\lambda)$ with all Gegenbauer coefficients zero, is the zero function. Applying this uniqueness the functions $f \star_{\lambda+1} g$ and $(2\lambda+1)\mathcal{D}((\mathcal{I}f)\star_\lambda(\mathcal{I}g))$ are equal almost everywhere on $[-1,1]$. □

5. Families of strictly positive definite functions constructed via convolution

In this section the convolution via the dimension hopping formula given in Theorem 4.1 is employed to generate a family of strictly positive definite zonal functions, essentially by the self-convolution of the characteristic functions of spherical caps.

There is a strong tradition in approximation theory of generating families of (strictly) positive definite functions by convolution. For example, univariate B-splines on a uniform mesh can be generated by repeated convolution of the characteristic function of an interval with itself. Further, in geostatistics, physical motivations give rise to the circular and spherical covariances. These are generated by convolving the characteristic function of a disk in $\mathbb{R}^2$, and of a ball in $\mathbb{R}^3$, with themselves. The Euclid hat functions, see Wu [20] and Gneiting [9], are a continuation of this method of construction beyond $\mathbb{R}^3$. Such a self-convolution will automatically have a nonnegative Fourier transform. There has also been considerable interest in the statistics community in the analogous self-convolution construction of kernels on the sphere. See, for example, Estrade and Istas [6], Tovchigrechko and Vakser [18] and Ziegel [22].

Let us consider this analogous construction for zonal functions on $\mathbb{S}^d$. Formula (iv) of Theorem 4.2 shows that all the Gegenbauer coefficients of the self-convolution

$$f = \chi_{[c,1]} \star_\lambda \chi_{[c,1]}, \quad -1 < c < 1,$$

are nonnegative so that $f \circ \cos$ is automatically positive definite. It remains to see if the self convolution of a spherical cap is strictly positive definite.

[1, (22.13.2)] gives the formula

$$\frac{n(2\alpha+n)}{2\alpha} \int_0^x C_n^\alpha(y) \left(1-y^2\right)^{\alpha-\frac{1}{2}} dy = C_{n-1}^{\alpha+1}(0) - \left(1-x^2\right)^{\alpha+\frac{1}{2}} C_{n-1}^{\alpha+1}(x),$$
$$\alpha > 0, \ n > 0,$$

from which it follows that

$$\frac{n(2\lambda+n)}{2\lambda} \int_c^1 C_n^\lambda(y) \left(1-y^2\right)^{\lambda-\frac{1}{2}} dy = \left(1-c^2\right)^{\lambda+\frac{1}{2}} C_{n-1}^{\lambda+1}(c), \quad \lambda > 0, n > 0. \tag{5.1}$$

The quantity on the left above is a positive multiple of the nth Gegenbauer coefficient of the characteristic function $\chi_{[c,1]}$. This corresponds to a spherical cap of radius $\arccos(c)$ in $\mathbb{S}^d$. Now if $\lambda, n > 0$ and $0 < c = \cos(s) < 1$ is a zero of $C_{n-1}^{\lambda+1}$, then the interlacing property of the zeros of $C_{n-1}^{(\beta)}$ and $C_n^{(\beta)}$ implies c is not a zero of $C_n^{\lambda+1}$. It follows from the three term recurrence relation for the Gegenbauer polynomials that c is also not a zero of $C_{n+1}^{\lambda+1}$. Hence, for $0 < c < 1$, $f = \chi_{[c,1]} \star_\lambda \chi_{[c,1]}$ has infinitely many Gegenbauer coefficients with respect to $\{C_n^\lambda\}$ of even index that are positive, and infinitely many coefficients of odd index that are positive.

Since it is clear that f is continuous, it follow that this function belongs to the cone $\Lambda_d^{\boxplus}$. In particular this shows that when $d \geq 2$, $f \in \Lambda_d^+$, that is that $f \circ \cos$ is a strictly positive definite zonal function on $\mathbb{S}^d$.

Let us now turn to the support of the function $f = \chi_{[c,1]} \star_\lambda \chi_{[c,1]}$. Here, the isomorphic convolution of zonal functions approach, summarized by Eqs. (4.6) and (4.7), comes into its own. In the setting of the sphere let both of the caps have angular radius r, $0 < r < \pi/2$. Let one of the caps have axis u, the north pole, and rotate the other cap with axis v away. Then it is clear that the caps no longer overlap if u and v are more than $2r$ apart. Thus, in the setting of the interval, and for $0 < c < 1$, $\chi_{[c,1]} \star_\lambda \chi_{[c,1]}$ has support $[\cos(2\arccos(c)), 1]$.

Let us apply this self-convolution approach to generate locally supported strictly positive definite functions on $\mathbb{S}^d$. The function for $\mathbb{S}^d$ has the form $N_{d,c} \circ \cos$ where

$$N_{d,c} = N_{2\lambda+1,c} = g_{\lambda,c}\, \chi_{[c,1]} \star_\lambda \chi_{[c,1]}, \tag{5.2}$$

and $\operatorname{supp} N_{d,c} = [\cos(2\arccos(c)), 1]$. Here $g_{\lambda,c}$ is a normalizing constant chosen so that $N_{d,c}(1) = 1$. As mentioned previously this echoes a well known method of construction of positive definite kernels for $\mathbb{R}^d$.

Fig. 1 gives plots of $N_{d,c}$ for various values of d and $c = \frac{1}{2}$.

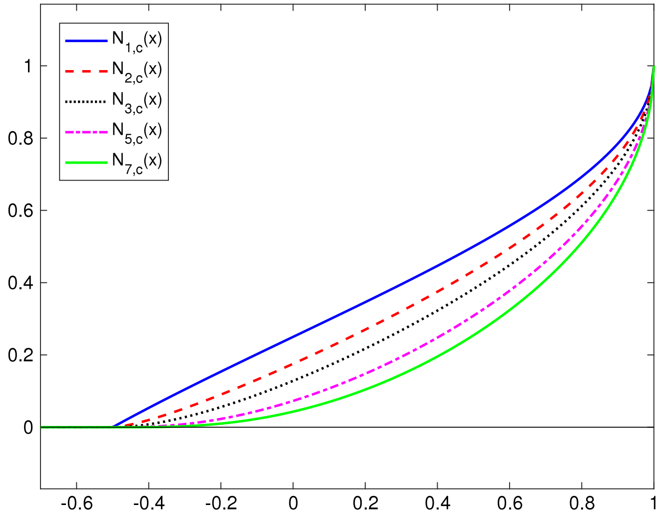

Fig. 1. The normalized self-convolutions $N_{d,c}$ for $c = 1/2$ and various values of d.

We will give a few more details of the calculation on $\mathbb{S}^3$. The desired function in Λ_3^+ is essentially to be obtained by convolving spherical caps. Thus, in the setting of the interval, we wish to calculate

$$f = g \star_1 g, \quad \text{with } g = \chi_{[c,1]} \text{ and } 0 < c < 1.$$

Employing the dimension hopping approach embodied in Theorem 4.1

$$g \star_1 g = \mathcal{D}\{(\mathcal{I}g) \star_0 (\mathcal{I}g)\}.$$

Now $(\mathcal{I}g)(x) = (x - c)_+^1$. Hence, for $0 < c = \cos s < 1$ and $0 < \theta < 2s$,

$$\begin{aligned}
&((\mathcal{I}g) \star_0 (\mathcal{I}g))(\cos\theta) \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (\mathcal{I}g)(\cos(\theta - t))(\mathcal{I}g)(\cos t)\, dt = \frac{1}{2}\int_{\theta-s}^{s} [\cos(\theta - t) - c][\cos(t) - c]\, dt \\
&= \frac{1}{4}(2s - \arccos(x))(x + \cos(2s) + 1) - \frac{1}{4}\sin(2s)x \\
&\quad + \frac{1}{4}(2 + \cos(2s))\sqrt{1 - x^2} - \frac{1}{2}\sin(2s),
\end{aligned}$$

where $x = \cos(\theta)$. Consequently, applying $\mathcal{D}$ and normalizing so that the function has value 1 at $x = 1$ we obtain the locally supported basis function $N_{3,c} \in \Lambda_3^+$,

$$N_{3,c}(x) = \begin{cases} 1 + b\arccos(x) + d\sqrt{\dfrac{1 - x}{1 + x}}, & \cos(2s) < x \le 1, \\ 0, & -1 \le x \le \cos(2s), \end{cases}$$

where $ab = -\frac{1}{4}$, $ad = \frac{1}{4}(1 + \cos(2s))$ and $a = (g \star_1 g)(1) = \frac{1}{2}s - \frac{1}{4}\sin(2s)$.

Similar arguments to those used in the indirect computation of $g \star_1 g$ show

$$g \star_2 g = 3\, \mathcal{D}^2 \left\{(\mathcal{I}^2 g) \star_0 (\mathcal{I}^2 g)\right\}, \qquad g \star_3 g = 15\, \mathcal{D}^3 \left\{(\mathcal{I}^3 g) \star_0 (\mathcal{I}^3 g)\right\}, \text{ etc.}$$

Carrying out the details of these calculations yields the normalized locally supported basis functions $N_{5,c} \in \Lambda_5^+$, $N_{7,c} \in \Lambda_7^+$ and $N_{9,c} \in \Lambda_9^+$ specified below.

$$N_{5,c}(x) = \begin{cases} 1 + b\arccos(x) + \sqrt{\dfrac{1-x}{1+x}}\left(d + \dfrac{e}{1+x}\right), & \cos(2s) < x \le 1, \\ 0, & -1 \le x \le \cos(2s), \end{cases}$$

where $ab = -\frac{3}{16}$, $ad = \frac{3}{4}\cos^2(s) - \frac{1}{4}\cos^4(s)$, $ae = -\frac{1}{4}\cos^4(s)$ and $a = (g \star_2 g)(1) = \frac{1}{4}\sin(s)\cos^3(s) - \frac{5}{8}\sin(s)\cos(s) + \frac{3}{8}s$.

$$N_{7,c}(x) = \begin{cases} 1 + b\arccos(x) + \sqrt{\dfrac{1-x}{1+x}}\left(d + \dfrac{e}{v} + \dfrac{f}{v^2}\right), & \cos(2s) < x \le 1, \\ 0, & -1 \le x \le \cos(2s), \end{cases}$$

where $v = 1 + x$, $ab = -\frac{5}{32}$, $ad = \frac{15}{16}\cos^2(s) - \frac{5}{8}\cos^4(s) + \frac{1}{6}\cos^6(s)$, $ae = -\frac{5}{8}\cos^4(s) + \frac{1}{6}\cos^6(s)$, $af = \frac{1}{4}\cos^6(s)$ and $a = (g \star_3 g)(1) = \frac{5}{16}s - \frac{11}{16}\sin(s)\cos(s) + \frac{13}{24}\sin(s)\cos^3(s) - \frac{1}{6}\sin(s)\cos^5(s)$.

$$N_{9,c}(x) = \begin{cases} 1 + b\arccos(x) + \sqrt{\dfrac{1-x}{1+x}}\left(d + \dfrac{e}{v} + \dfrac{f}{v^2} + \dfrac{h}{v^3}\right), & \cos(2s) < x \le 1, \\ 0, & -1 \le x \le \cos(2s), \end{cases}$$

where $v = 1+x$, $ab = -\frac{35}{256}$, $ad = \left(105\cos^2(s) - 105\cos^4(s) + 56\cos^6(s) - 12\cos^8(s)\right)/96$, $ae = \left(-105\cos^4(s) + 56\cos^6(s) - 12\cos^8(s)\right)/96$, $af = \left(84\cos^6(s) - 18\cos^8(s)\right)/96$ $ah = -30\cos^8(s)/96$ and

$$\begin{aligned} a &= (g \star_4 g)(1) \\ &= \frac{35}{128}s - \frac{93}{128}\sin(s)\cos(s) + \frac{163}{192}\sin(s)\cos^3(s) \\ &\quad - \frac{25}{48}\sin(s)\cos^5(s) + \frac{1}{8}\sin(s)\cos^7(s). \end{aligned}$$

Clearly the functions $N_{3,c}$, $N_{5,c}$, $N_{7,c}$ and $N_{9,c}$ can be evaluated efficiently by precomputing all the coefficients, once and for all, and then using nested multiplication.

Some other members of the family of self-convolutions specified by (5.2) have been computed by other means. The function $N_{1,c}$ is easily computed directly from the definition of the one dimensional convolution (4.5). Written in terms of angles, $\theta = \arccos(x)$, it is a linear B-spline. In our setting of working predominately on $[-1, 1]$ it is for $0 < c = \cos(s) < 1$,

$$N_{1,c}(x) = \begin{cases} 0, & -1 \le x \le \cos(2s), \\ 1 - \dfrac{1}{2s}\arccos(x), & \cos(2s) \le x \le 1. \end{cases}$$

Furthermore, the function $N_{2,c}$ has been computed by Tovchigrechko and Vakser [18] using spherical trigonometry. $N_{2,c}(x)$ equals

$$\begin{cases} 0, & -1 \le x \le \cos(2s), \\ \dfrac{1}{2\pi(1-c)} \left\{ 2\pi - 2\arccos\left(\dfrac{x-c^2}{1-c^2}\right) \right. & \\ \left. \quad -4c\arccos\left(\dfrac{\cot(s)(1-x)}{\sin(\arccos(x))}\right)\right\}, & \cos(2s) < x < 1, \\ 1, & x = 1. \end{cases}$$

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